A NEW MEASUREMENT FOR NETWORK SHARING FAIRNESS

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Abstract: A quantitative measure for the fairness of network resource distribution was proposed in Jain, Chiu and Hawe [1]. This fairness function is widely adopted in network design and management. This paper proposes a new distribution fairness score function. Compared the proposed fairness function to the one by Jain, Chiu and Hawe, the proposed fairness function keeps all the nice properties that Jain, Chiu and Hawe′s fairness function possesses. In addition, the proposed fairness score function has better performance in dealing with completely unfair cases.

Key Words: Network, Network Performance Evaluation, Resource Distribution, Fairness, Fairness Score Function

I. INTRODUCTION

Network congestion control is a popular topic in network design and management. An important criterion to evaluate network performance is the resource allocation fairness. A quantitative measure to evaluate network resource sharing fairness was proposed in Jain, Chiu and Hawe [1]. Some other references can be found from literature. For examples, Bertsekas and Gallager [2], Chiu and Jain [3], Kelly et al. [4], Mazumdar, Mason and Douligeris [5]. Suppose the entire network resource is shared by n users. Let $x_1, x_2, \cdots, x_n$ be the amounts of resource n users receive. A quantitative measure is needed to evaluate the fairness of the network resource distribution. This quantitative measure is then a function of $x_1, x_2, \cdots, x_n$. The function should possess the following characteristics: If the distribution is completely unfair, i.e., only one of the n users occupies the entire resource, the value of the function should be 0; if the distribution is perfectly fair, i.e., all n users share the entire resource equally, the value of the function should be 1; when the distribution becomes fairer, the value of the function should increase. One of the widely adopted quantitative measures of the distribution fairness was defined by Jain, Chiu, and Hawe [1] as follows:

$$F(x_1, x_2, \cdots, x_n) = \frac{n^{1-\rho}\left(\sum_{i=1}^{n} x_i\right)^{\rho}}{\sum_{i=1}^{n} x_i^{\rho}} \qquad (\rho > 1) \tag{1}$$

where $x_1, x_2, \cdots, x_n$ are the amounts of resource n users receive. When ρ=2, it becomes

$$F(x_1, x_2, \cdots, x_n) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2}}{n\left(\sum_{i=1}^{n} x_i^{2}\right)} \tag{2}$$

Chiu and Jain [3] showed that $F(x_1, x_2, \cdots, x_n)$ defined in (2) possesses the following properties:

(a) $0 \le F(x_1, x_2, \cdots, x_n) \le 1$ for any nonnegative $x_1, x_2, \cdots, x_n$.

(b) $F(x_1, x_2, \cdots, x_n) = 1/n$ when the distribution is completely unfair, i.e., only one user occupies the entire resource while the other users do not receive any. $F(x_1, x_2, \cdots, x_n) = 1$ when the distribution is perfectly fair, i.e., all n users share the entire resource equally.

(c) The fairness score does not depend on scale.

(d) The fairness score function continuously reflects changes in allocation.

(e) If only k out of n users share the entire resource equally while the others do not receive any, then the fairness score is k/n.

Properties (a), (c) and (d) are attractive to the researchers and users. However, the result that $F(x_1, x_2, \cdots, x_n) = 1/n$ for the completely unfair case does not fit the real situation well. If only one user occupies the entire network resource, the value of the function, $F(x, 0, \cdots, 0)$, should be zero. Property (e) should be modified for the same reason. Actually, when k=1, the same problem will occur because it is the completely unfair situation. The fairness score should be zero, not 1/n.

In this paper, a new fairness function, $G(x_1, x_2, \cdots, x_n)$, is proposed. The proposed fairness score function possesses properties (a), (c) and (d) mentioned above. It satisfies all the conditions in (b) except the one for completely unfair case $F(x_1, x_2, \cdots, x_n) = 1/n$. In fact, if $G(x_1, x_2, \cdots, x_n)$ is used, the value of the fairness score is zero for the completely unfair case. For the case that only k out of n users share the entire resource equally, the value of the fairness score based on the proposed fairness score function is

$$\frac{n(k-1)}{k(n-1)}.$$

If k=1, that is the case of completely unfair sharing. The fairness score for that case is zero. Thus it can be concluded that $G(x_1, x_2, \cdots, x_n)$ better fits the real world situation than $F(x_1, x_2, \cdots, x_n)$ does.

II. A NEW FAIRNESS SCORE FUNCTION

Define fairness function $G(x_1, x_2, \cdots, x_n)$ as follows:

$$G(x_1, x_2, \cdots, x_n) = 1 - \frac{n \sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{1}{n} \right]^2}{n-1} . \quad (3)$$

Theorem 1. For any $x_1 \geq 0, \ldots, x_n \geq 0$,

$$0 \leq G(x_1, x_2, \cdots, x_n) \leq 1 . \quad (4)$$

Proof. For any $x_1 \geq 0, \ldots, x_n \geq 0$,

$$\sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{1}{n} \right]^2 = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum_{j=1}^{n} x_j \right)^2} - \frac{2 \sum_{i=1}^{n} x_i}{n \left(\sum_{j=1}^{n} x_j \right)} + \frac{1}{n} = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum_{j=1}^{n} x_j \right)^2} - \frac{1}{n} \leq 1 - \frac{1}{n}.$$

This is because

$$\sum_{i=1}^{n} x_i^2 \leq \left(\sum_{j=1}^{n} x_j \right)^2$$

for any nonnegative $x_1, x_2, \cdots, x_n$. Then

$$G(x_1, x_2, \cdots, x_n) \geq 1 - \frac{n\left(1 - \frac{1}{n}\right)}{n-1} = 0.$$

$G(x_1, x_2, \cdots, x_n) \leq 1$ is obvious. ▌

Theorem 2. $G(x_1, x_2, \cdots, x_n) = 1$ if and only if the distribution is perfectly fair. Here perfectly fair distribution refers to the case that all n user share the entire network resource equally.

Proof. $G(x_1, x_2, \cdots, x_n) = 1$ if and if

$$\sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{1}{n} \right]^2 = 0. \tag{5}$$

Statement (5) is true if and only if

$$\frac{x_1}{\sum_{j=1}^{n} x_j} - \frac{1}{n} = \frac{x_2}{\sum_{j=1}^{n} x_j} - \frac{1}{n} = \cdots = \frac{x_n}{\sum_{j=1}^{n} x_j} - \frac{1}{n} = 0.$$

It is equivalent to

$$\frac{x_1}{\sum_{j=1}^{n} x_j} = \frac{x_2}{\sum_{j=1}^{n} x_j} = \cdots = \frac{x_n}{\sum_{j=1}^{n} x_j} = \frac{1}{n}.$$

It follows that

$$x_1 = x_2 = \cdots = x_n.$$

This ends the proof of the Theorem. ▌

Theorem 3. $G(x_1, x_2, \cdots, x_n) = 0$ if and only if the distribution is completely unfair. Here the case that the distribution is completely unfair is the one that only one user occupies the entire resource.

Proof. When only one user occupies the entire resource.

$$G(x_1, x_2, \cdots, x_n) = G(x, 0, \cdots, 0) = 1 - \frac{n\left[\left(\frac{x}{x} - \frac{1}{n}\right)^2 + (n-1)\left(\frac{0}{x} - \frac{1}{n}\right)^2\right]}{n-1} = 0 .$$

Now it is desired to show that if $G(x_1, x_2, \cdots, x_n) = 0$, then it is the case that only one of the n users occupies the entire resource. To show this, define

$$a_i = \frac{x_i}{\sum_{j=1}^{n} x_j} \qquad i = 1, 2, \cdots, n .$$

Then

$$G(x_1, x_2, \cdots, x_n) = 1 - \frac{n \sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{1}{n}\right]^2}{n-1} = 1 - \frac{n \sum_{i=1}^{n} \left[a_i - \frac{1}{n}\right]^2}{n-1}$$

$$= 1 - \frac{n \sum_{i=1}^{n} a_i^2 - 1}{n-1} .$$

If $G(x_1, x_2, \cdots, x_n) = 0$, then

$$\frac{n \sum_{i=1}^{n} a_i^2 - 1}{n-1} = 1 .$$

This implies

$$\sum_{i=1}^{n} a_i^2 = 1 .$$

Note that $a_1, \ldots, a_n$ are nonnegative and that

$$\sum_{i=1}^{n} a_i = 1 .$$

Then

$$\sum_{i=1}^{n} a_i^2 \le \left(\sum_{i=1}^{n} a_i\right)^2 = 1 .$$

The only case that validates

$$\sum_{i=1}^{n} a_i^2 = 1$$

is the one that one of $a_1, \ldots, a_n$ is 1 and the rest are 0. It completes the proof. ▌

Theorem 4. If only k out of n users share the entire resource equally while the other n–k users do not share any, then

$$G(x_1, x_2, \cdots, x_n) = \frac{n(k-1)}{k(n-1)} .$$

Proof. When only k out of n users share the entire resource equally,

$$G(x_1, x_2, \cdots, x_n) = G\Big(\underbrace{x, \cdots, x}_{k}, \underbrace{0, \cdots, 0}_{n-k}\Big) = 1 - \frac{n\left\{\sum_{i=1}^{k}\left[\frac{1}{k} - \frac{1}{n}\right]^2 + \sum_{i=k+1}^{n}\left[0 - \frac{1}{n}\right]^2\right\}}{n-1}$$

$$= 1 - \frac{n\left[\frac{(n-k)^2}{kn^2} + \frac{n-k}{n^2}\right]}{n-1} = 1 - \frac{n \cdot \frac{n-k}{nk}}{n-1} = \frac{n(k-1)}{k(n-1)} .$$ ▌

Note that for fixed n,

$$\frac{n(k-1)}{k(n-1)}$$

is increasing in k. This can be seen by rewriting $G(x_1, x_2, \cdots, x_n)$ as

$$\frac{1 - \frac{1}{k}}{1 - \frac{1}{n}} .$$

This result is consistent with the fact that the distribution will become fairer if more users share the entire resource. It should also be noted that this result makes more sense than the one using $F(x_1, x_2, \cdots, x_n)$. An example to show this is to take n=2 and k=1. It means that one of the two users occupies the whole resource while the other does not receive any. When the fairness score function $F(x_1, x_2, \cdots, x_n)$ is used,

$$F(x, 0) = \frac{1}{2}.$$

It does not meet the real situation because the case is a completely unfair distribution and the fairness score should be zero. On the other hand, when the fairness score function $G(x_1, x_2, \cdots, x_n)$ is used,

$$G(x, 0) = 0.$$

This is the result which is expected for completely unfair resource allocation.

The following result shows that the value of $G(x_1, x_2, \cdots, x_n)$ increases when the distribution becomes more and more equally.

Theorem 5. For $\eta>0$, define

$$D(\eta) = G(x_1, \cdots, x_s - \eta, \cdots, x_t + \eta, \cdots, x_n) - G(x_1, \cdots, x_s, \cdots, x_t, \cdots, x_n) .$$

Then

$$D(\eta) \begin{cases} > 0 & \text{if } \eta > x_s - x_t \\ = 0 & \text{if } \eta = x_s - x_t \\ < 0 & \text{if } \eta < x_s - x_t \end{cases}.$$

Proof. By the definition of $D(\eta)$,

$$D(\eta)=\frac{n\left[\left(\frac{x_s}{\sum_{j=1}^{n}x_j}-\frac{1}{n}\right)^2+\left(\frac{x_t}{\sum_{j=1}^{n}x_j}-\frac{1}{n}\right)^2-\left(\frac{x_s-\eta}{\sum_{j=1}^{n}x_j}-\frac{1}{n}\right)^2-\left(\frac{x_t+\eta}{\sum_{j=1}^{n}x_j}-\frac{1}{n}\right)^2\right]}{n-1}$$

$$=\frac{\left(nx_s-\sum_{j=1}^{n}x_j\right)^2+\left(nx_t-\sum_{j=1}^{n}x_j\right)^2-\left(nx_s-n\eta-\sum_{j=1}^{n}x_j\right)^2-\left(nx_t+n\eta-\sum_{j=1}^{n}x_j\right)^2}{n(n-1)\left(\sum_{j=1}^{n}x_j\right)^2}$$

$$=\frac{n\eta\left(2nx_s-n\eta-2\sum_{j=1}^{n}x_j\right)-n\eta\left(2nx_t-n\eta-2\sum_{j=1}^{n}x_j\right)}{n(n-1)\left(\sum_{j=1}^{n}x_j\right)^2}$$

$$=\frac{2n^2\eta\left(x_s-x_t-\eta\right)}{n(n-1)\left(\sum_{j=1}^{n}x_j\right)^2}.$$

The proof then follows. ▌

The following result shows that if all n users are given the same extra amount of network resource, the fairness of the distribution will not decrease.

Theorem 6. $G(x_1+\delta, x_2+\delta, \cdots, x_n+\delta) \geq G(x_1, x_2, \cdots, x_n)$ for any $\delta>0$.

Proof. It can be shown that $G(x_1+\delta, x_2+\delta, \cdots, x_n+\delta)$ is an increasing function of δ. In fact,

$$G(x_1+\delta, x_2+\delta, \cdots, x_n+\delta) = 1 - \frac{n\sum_{i=1}^{n}\left[\frac{x_i+\delta}{\sum_{j=1}^{n} x_j + n\delta} - \frac{1}{n}\right]^2}{n-1}$$

$$= 1 - \frac{n\sum_{i=1}^{n}\left[\frac{n x_i - \sum_{j=1}^{n} x_j}{n\left(\sum_{j=1}^{n} x_j + n\delta\right)}\right]^2}{n-1}$$

$$= 1 - \frac{\sum_{i=1}^{n}\left(n x_i - \sum_{j=1}^{n} x_j\right)^2}{n(n-1)\left(\sum_{j=1}^{n} x_j + n\delta\right)^2}.$$

Then $G(x_1+\delta, x_2+\delta, \cdots, x_n+\delta)$ increases in δ. The proof follows. ▌

Example In the case that there are two users (n=2),

$$G(x_1, x_2) = 1 - \frac{2\times\left[\left(\frac{x_1}{x_1+x_2} - \frac{1}{2}\right)^2 + \left(\frac{x_2}{x_1+x_2} - \frac{1}{2}\right)^2\right]}{2-1} = \frac{4x_1x_2}{(x_1+x_2)^2}.$$

III. CONCLUSION

A new network sharing fairness function, $G(x_1, x_2, \cdots, x_n)$ is proposed in this paper. Comparing the proposed fairness function to the widely adopted fairness function $F(x_1, x_2, \cdots, x_n)$ proposed by Jain, Chiu, and Hawe [1], the proposed fairness function in this paper possesses all the meritorious properties that Jain, Chiu, and Hawe′s fairness function does. In addition, $G(x_1, x_2, \cdots, x_n)$ performs better than $F(x_1, x_2, \cdots, x_n)$ does for the completely unfair case.

The fairness function can be used to evaluate the fairness of the resource distribution. When it is found that the distribution of the network resource is significantly unfair, some action needs to be taken. As discussed in Section II, the proposed fairness score function $G(x_1, x_2, \cdots, x_n)$ can be rewritten as:

$$G(x_1, x_2, \cdots, x_n) = 1 - \frac{n \sum_{i=1}^{n} a_i^2 - 1}{n - 1}$$

where

$$a_i = \frac{x_i}{\sum_{j=1}^{n} x_j} \qquad i = 1, 2, \cdots, n .$$

It can be seen that the distribution of $G(x_1, x_2, \cdots, x_n)$ is scale-free. Thus upper percentiles of $G(x_1, x_2, \cdots, x_n)$ can be obtained by Monte-Carlo simulation. A statistical test can then be conducted to check whether or not the network resource distribution is statistically significantly unfair.

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