NETWORK SHARING FAIRNESS FUNCTION WITH CONSIDERATION OF PRIORITY LEVELS

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Abstract

An important issue of computer network management is to distribute network resource fairly to its users. The first quantitative fairness score function $F(x_1, x_2, \cdots, x_n)$ was proposed in 1984 by Jain, Chiu, and Hawe [1] for evaluating network resource sharing fairness. Chen and Zhang [6] proposed another fairness score function $G(x_1, x_2, \cdots, x_n)$ which better fits the real world situation in some cases. This paper considers the situation that the users can have different priority levels. A modified fairness score function $G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n)$ is proposed. The proposed fairness score function can deal with the situation that the network resource users have different priority levels, and can keep all the nice properties of $G(x_1, x_2, \cdots, x_n)$.

Key Words: Network, Network Performance Evaluation, Resource Distribution, Fairness, Fairness Score Function, Priority Level

I. INTRODUCTION

One of the important issues of computer network management is to distribute network resource fairly to its users. Numerous research papers have been published in this area in the literature. To quantify the fairness of the network resource distribution, Jain, Chiu and Hawe [1] proposed a quantitative measure for evaluating network resource sharing fairness. For more references in this field, see Bertsekas and Gallager [2], Chiu and Jain [3], Kelly et al. [4], Mazumdar, Mason and Douligeris [5]. Suppose the entire network resource is shared by n users. Let $x_1, x_2, \cdots, x_n$ be the amounts of resource the users receive respectively. The fairness score function proposed by Jain, Chiu, and Hawe [1] is:

$$F(x_1, x_2, \cdots, x_n) = \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n\left(\sum_{i=1}^{n} x_i^2\right)} \tag{1}$$

This fairness score function $F(x_1, x_2, \cdots, x_n)$ possesses some properties. It can be shown easily that $0 \le F(x_1, x_2, \cdots, x_n) \le 1$ for any nonnegative $x_1, x_2, \cdots, x_n$. In the case that the distribution is completely unfair, i.e., only one user occupies the entire resource while the other users do not receive any, the value of $F(x_1, x_2, \cdots, x_n)$ is $1/n$. On the other hand, if the distribution is perfectly fair, i.e., all the users share the entire resource equally, then the value of $F(x_1, x_2, \cdots, x_n)$ is 1. If only *k* out of *n* users share the entire resource equally while the others do not receive any, then the fairness score is k/n. It can be seen that $F(x_1, x_2, \cdots, x_n)$ does not depend on scale. It can also be seen that this fairness score function continuously reflects changes in allocation. The result that $F(x_1, x_2, \cdots, x_n) = 1/n$ for the completely unfair case does not fit the real situation well. In fact, if only one user occupies the entire network resource, the value of the function, $F(x, 0, \cdots, 0)$, should be zero, not $1/n$. The same thing happens to the *k*-out-of-*n* case. When $k = 1$, the same problem will occur because it is the completely unfair situation. The fairness score should be zero, not $1/n$. Chen and Zhang [6] proposed another fairness score function:

$$G(x_1, x_2, \cdots, x_n) = 1 - \frac{n \sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{1}{n} \right]^2}{n-1}. \tag{2}$$

It has been shown that $G(x_1, x_2, \cdots, x_n)$ keeps all the nice properties mentioned above. For the completely unfair case,

$$G(x_1, x_2, \cdots, x_n) = G(x, 0, \cdots, 0) = 0.$$

For the case that only *k* out of *n* users share the entire resource equally, the value of the fairness score based on the fairness score function in (2) is

$$\frac{n(k-1)}{k(n-1)}.$$

If $k = 1$, that is the case of completely unfair sharing. The fairness score for that case is zero.

In this paper, the fairness score function $G(x_1, x_2, \cdots, x_n)$ is modified to accommodate the situation that the users are at different priority levels. Equally distributing network resource to all the users is actually unfair if the users are at different priority levels. Instead, the system should distribute the network resource to the uses proportionally according to their priority levels. For example, in a scenario where users pay different prices for their bandwidths, the weights in the fairness metric should be assigned in proportion to the bandwidth allocation. It will be shown in the next section that the fairness function $G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n)$ proposed in this paper keeps all the meritorious properties of $G(x_1, x_2, \cdots, x_n)$ even for the case that the users are at different priority levels.

II. FAIRNESS SCORE FUNCTION CONSIDERING PRIORITY LEVELS

Let $x_1, x_2, \cdots, x_n$ be the amounts of resource that the users receive respectively. Also let $w_1, w_2, \cdots, w_n$ be the corresponding priority factors of these users. It means that if the amount of resource which a basic user receives is *x*, then user *i* is supposed to receive $w_i x \ \left(i = 1, 2, \cdots, n\right)$.

Define the fairness score function $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right)$ as follows:

$$G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) = 1 - \frac{\left(\sum_{i=1}^{n} w_i\right)^2 \cdot \sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2}{\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2} . \tag{3}$$

In the case that all the users are at the same priority level, i.e.,

$$w_1 = w_2 = \cdots = w_n ,$$

the fairness score function $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right)$ becomes $G\left(x_1, x_2, \cdots, x_n\right)$ which is defined in (2). It will be shown that the fairness score function $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right)$ possesses the same meritorious properties as $G\left(x_1, x_2, \cdots, x_n\right)$ does. The following result shows that the value of $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right)$ is always between 0 and 1.

Theorem 1. For any $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$,

$$0 \leq G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) \leq 1 . \tag{4}$$

Proof. $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) \leq 1$ is obvious. To show

$$G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) \geq 0 ,$$

sort $w_1, w_2, \cdots, w_n$ from the smallest to the largest. Let $w_{(i)}$ be the ith smallest number $(i = 1, 2, \cdots, n)$. Without loss of generality, it can be assumed that

$$w_1 = \min\{ w_1, w_2, \cdots, w_n \}.$$

Thus $w_1 = w_{(1)}$. Then

$$\sum_{i=1}^{n}\left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2 = \left[\frac{x_1}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right]^2 + \sum_{i=2}^{n}\left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2$$

$$= \left[\frac{x_1}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right]^2 + \sum_{i=2}^{n}\left[\left(\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right) + \left(\frac{w_1}{\sum_{j=1}^{n} w_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right)\right]^2$$

$$= \left[\frac{x_1}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right]^2 + \sum_{i=2}^{n}\left(\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right)^2 + \sum_{i=2}^{n}\left(\frac{w_1}{\sum_{j=1}^{n} w_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right)^2$$

$$+2\sum_{i=2}^{n}\left(\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right)\left(\frac{w_1}{\sum_{j=1}^{n} w_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right)^2 + \sum_{i=2}^{n}\left(\frac{w_1}{\sum_{j=1}^{n} w_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right)\left(\frac{2x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1 + w_i}{\sum_{j=1}^{n} w_j}\right).$$

Define

$$P = \sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j}\right)^2 \tag{5}$$

and

$$Q=\sum_{i=2}^{n}\left(\frac{w_1}{\sum_{j=1}^{n}w_j}-\frac{w_i}{\sum_{j=1}^{n}w_j}\right)\left(\frac{2x_i}{\sum_{j=1}^{n}x_j}-\frac{w_1+w_i}{\sum_{j=1}^{n}w_j}\right). \tag{6}$$

Note that

$$P=\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}\right)^2-\frac{2w_1}{\sum_{j=1}^{n}w_j}\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}\right)+\sum_{i=1}^{n}\left(\frac{w_1}{\sum_{j=1}^{n}w_j}\right)^2=\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}\right)^2-\frac{2w_1}{\sum_{j=1}^{n}w_j}+\sum_{i=1}^{n}\left(\frac{w_1}{\sum_{j=1}^{n}w_j}\right)^2.$$

Since $w_1, w_2, \cdots, w_n$ are fixed numbers, then to maximize *P*, it is desired to maximize

$$R=\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}\right)^2$$

Since

$$\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}\right)=1,$$

then *R* is maximized if and only if one of the

$$\frac{x_i}{\sum_{j=1}^{n}x_j}\quad(i=1,\cdots,n)$$

values is one, and the rest are zeros. Since

$$\frac{w_1}{\sum_{j=1}^{n}w_j}-\frac{w_i}{\sum_{j=1}^{n}w_j}\le 0\quad(i=2,3,\cdots,n),$$

then to maximize Q ,

$$\frac{2x_i}{\sum_{j=1}^{n} x_j} = 0$$

must be true for those term satisfying $w_1 < w_i$ for $i = 2,3,\cdots,n$. It is equivalent to the condition that x_i must be 0 for those terms satisfying $w_1 < w_i$ for $i = 2,3,\cdots,n$. It means that the amounts that are received by users, whose priority levels are not the lowest, must be zero.

Combining the above facts,

$$\sum_{i=1}^{n}\left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2 \le \left[1 - \frac{w_{(1)}}{\sum_{j=1}^{n} w_j}\right]^2 + \sum_{i=2}^{n}\left[0 - \frac{w_{(i)}}{\sum_{j=1}^{n} w_j}\right]^2 = \left[1 - \frac{w_{(1)}}{\sum_{j=1}^{n} w_j}\right]^2 + \sum_{i=2}^{n}\left[0 - \frac{w_{(i)}}{\sum_{j=1}^{n} w_j}\right]^2$$

$$= \frac{\left(\sum_{j=1}^{n} w_j - w_{(1)}\right)^2}{\left(\sum_{j=1}^{n} w_j\right)^2} + \frac{\sum_{i=2}^{n} w_{(i)}^2}{\left(\sum_{j=1}^{n} w_j\right)^2} = \frac{\left(\sum_{j=1}^{n} w_j - w_{(1)}\right)^2}{\left(\sum_{j=1}^{n} w_j\right)^2} + \frac{\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2}{\left(\sum_{j=1}^{n} w_j\right)^2}.$$

This implies that

$$0 \le G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) \le 1. \blacksquare$$

Theorem 2. $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) = 1$ if and only if the distribution is perfectly fair. Here perfectly fair distribution refers to the case that all the users share the entire network resource proportionally according to their priority levels.

Proof. By the definition of $G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right)$,

$$G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right) = 1$$

if and if

$$\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}-\frac{w_i}{\sum_{j=1}^{n}w_j}\right)^2=0. \tag{7}$$

This statement is true if and only if

$$\frac{x_1}{\sum_{j=1}^{n}x_j}-\frac{w_1}{\sum_{j=1}^{n}w_j}=\frac{x_2}{\sum_{j=1}^{n}x_j}-\frac{w_2}{\sum_{j=1}^{n}w_j}=\cdots=\frac{x_n}{\sum_{j=1}^{n}x_j}-\frac{w_n}{\sum_{j=1}^{n}w_j}=0\,.$$

It is equivalent to

$$\frac{x_1}{\sum_{j=1}^{n}x_j}=\frac{w_1}{\sum_{j=1}^{n}w_j},\ \frac{x_2}{\sum_{j=1}^{n}x_j}=\frac{w_2}{\sum_{j=1}^{n}w_j},\ \cdots,\ \frac{x_n}{\sum_{j=1}^{n}x_j}=\frac{w_n}{\sum_{j=1}^{n}w_j}\,.$$

This ends the proof of the Theorem. ▌

Theorem 3. $G^*\left(x_1,x_2,\cdots,x_n;\,w_1,w_2,\cdots,w_n\right)=0$ if and only if the distribution is completely unfair.

Here the case that the distribution is completely unfair is the one that only one user with the lowest priority level occupies the entire resource.

Proof. If only one user with the lowest priority level occupies the entire resource, then

$$G^*\left(x_1,x_2,\cdots,x_n;\,w_1,w_2,\cdots,w_n\right)=1-\frac{\left(\sum_{i=1}^{n}w_i\right)^2\cdot\left\{\left[1-\frac{w_{(1)}}{\sum_{j=1}^{n}w_j}\right]^2+\sum_{i=2}^{n}\left[0-\frac{w_{(i)}}{\sum_{j=1}^{n}w_j}\right]^2\right\}}{\left(\sum_{i=2}^{n}w_{(i)}\right)^2+\sum_{i=2}^{n}w_{(i)}^2}$$

$$=1-\frac{\left(\sum_{i=1}^{n} w_i\right)^2 \cdot \left\{ \frac{\left(\sum_{j=1}^{n} w_j - w_{(1)}\right)^2}{\left(\sum_{j=1}^{n} w_j\right)^2} + \frac{\sum_{i=2}^{n} w_{(i)}^2}{\left(\sum_{j=1}^{n} w_j\right)^2} \right\}}{\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2} = 1-\frac{\left(\sum_{j=1}^{n} w_j - w_{(1)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2}{\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2}$$

$$=1-\frac{\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2}{\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2} = 0.$$

Now it is desired to show that if $G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n) = 0$, then it is the case that only one of the users with the lowest priority level occupies the entire resource. To show this, note that

$$G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n) \geq 0 .$$

To let $G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n)$ reach 0, it is necessary to minimize

$$G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n) .$$

Since $w_1, w_2, \cdots, w_n$ are fixed numbers, then it is necessary to maximize

$$\sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_i}{\sum_{j=1}^{n} w_j} \right]^2 .$$

It has been mentioned above that

$$\sum_{i=1}^{n} \left[\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_i}{\sum_{j=1}^{n} w_j} \right]^2 = \sum_{i=1}^{n} \left(\frac{x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1}{\sum_{j=1}^{n} w_j} \right)^2 + \sum_{i=2}^{n} \left(\frac{w_1}{\sum_{j=1}^{n} w_j} - \frac{w_i}{\sum_{j=1}^{n} w_j} \right) \left(\frac{2x_i}{\sum_{j=1}^{n} x_j} - \frac{w_1 + w_i}{\sum_{j=1}^{n} w_j} \right) .$$

Following the same steps in the proof of Theorem 1, it can be shown that the only case that can minimize both

$$\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{n}x_j}-\frac{w_1}{\sum_{j=1}^{n}w_j}\right)^2$$

and

$$\sum_{i=2}^{n}\left(\frac{w_1}{\sum_{j=1}^{n}w_j}-\frac{w_i}{\sum_{j=1}^{n}w_j}\right)\left(\frac{2x_i}{\sum_{j=1}^{n}x_j}-\frac{w_1+w_i}{\sum_{j=1}^{n}w_j}\right)$$

is the case that only one of the users with the lowest priority level occupies the entire resource. It completes the proof. ▌

Theorem 4. If all the *n* users are at the same priority level, and if only *k* out of the n users share the entire resource equally while the other $n-k$ users do not share any, then

$$G^*\left(x_1,x_2,\cdots,x_n;\, w,w,\cdots,w\right) = \frac{n(k-1)}{k(n-1)}. \quad ▌$$

The proof of this theorem is obvious.

Theorem 5. For $\eta > 0$, define

$$\begin{aligned} D^*(\eta) = {} & G^*\left(x_1,\cdots,x_s-\eta,\,\cdots,\,x_t+\eta,\cdots,x_n;\, w_1,w_2,\cdots,w_n\right) \\ & -G^*\left(x_1,\cdots,x_s,\cdots,x_t,\cdots,x_n;\, w_1,w_2,\cdots,w_n\right). \end{aligned} \tag{8}$$

Then

$$D^*(\eta)\begin{cases} >0 & if\ \eta < \left[\left(\dfrac{x_s}{\sum_{j=1}^n x_j}-\dfrac{w_s}{\sum_{j=1}^n w_j}\right)-\left(\dfrac{x_t}{\sum_{j=1}^n x_j}-\dfrac{w_t}{\sum_{j=1}^n w_j}\right)\right]\Big/\left(\sum_{j=1}^n x_j\right) \\ =0 & if\ \eta = \left[\left(\dfrac{x_s}{\sum_{j=1}^n x_j}-\dfrac{w_s}{\sum_{j=1}^n w_j}\right)-\left(\dfrac{x_t}{\sum_{j=1}^n x_j}-\dfrac{w_t}{\sum_{j=1}^n w_j}\right)\right]\Big/\left(\sum_{j=1}^n x_j\right) \\ <0 & if\ \eta > \left[\left(\dfrac{x_s}{\sum_{j=1}^n x_j}-\dfrac{w_s}{\sum_{j=1}^n w_j}\right)-\left(\dfrac{x_t}{\sum_{j=1}^n x_j}-\dfrac{w_t}{\sum_{j=1}^n w_j}\right)\right]\Big/\left(\sum_{j=1}^n x_j\right) \end{cases} \tag{9}$$

Proof. By the definition of $D(\eta)$,

$$D^*(\eta)=\frac{\left(\sum_{i=1}^n w_i\right)^2\cdot\left[\left(\dfrac{x_s}{\sum_{j=1}^n x_j}-\dfrac{w_s}{\sum_{j=1}^n w_j}\right)^2+\left(\dfrac{x_t}{\sum_{j=1}^n x_j}-\dfrac{w_t}{\sum_{j=1}^n w_j}\right)^2-\left(\dfrac{x_s-\eta}{\sum_{j=1}^n x_j}-\dfrac{w_s}{\sum_{j=1}^n w_j}\right)^2-\left(\dfrac{x_t-\eta}{\sum_{j=1}^n x_j}-\dfrac{w_t}{\sum_{j=1}^n w_j}\right)^2\right]}{\left(\sum_{i=2}^n w_{(i)}\right)^2+\sum_{i=2}^n w_{(i)}^2}$$

$$=\frac{\left(\sum_{i=1}^n w_i\right)^2\cdot\left[\left(\dfrac{2x_s-\eta}{\sum_{j=1}^n x_j}-\dfrac{2w_s}{\sum_{j=1}^n w_j}\right)\left(\dfrac{\eta}{\sum_{j=1}^n x_j}\right)+\left(\dfrac{2x_t+\eta}{\sum_{j=1}^n x_j}-\dfrac{2w_t}{\sum_{j=1}^n w_j}\right)\left(-\dfrac{\eta}{\sum_{j=1}^n x_j}\right)\right]}{\left(\sum_{i=2}^n w_{(i)}\right)^2+\sum_{i=2}^n w_{(i)}^2}$$

$$=\frac{\left(\sum_{i=1}^n w_i\right)^2\cdot\eta\cdot\left[\left(\dfrac{2x_s-\eta}{\sum_{j=1}^n x_j}-\dfrac{2w_s}{\sum_{j=1}^n w_j}\right)-\left(\dfrac{2x_t+\eta}{\sum_{j=1}^n x_j}-\dfrac{2w_t}{\sum_{j=1}^n w_j}\right)\right]}{\left(\left(\sum_{i=2}^n w_{(i)}\right)^2+\sum_{i=2}^n w_{(i)}^2\right)\left(\sum_{j=1}^n x_j\right)}$$

$$= \frac{\left(\sum_{i=1}^{n} w_i\right)^2 \cdot (2\eta) \cdot \left(\frac{x_s - x_t - \eta}{\sum_{j=1}^{n} x_j} + \frac{w_t - w_s}{\sum_{j=1}^{n} w_j} \right)}{\left(\left(\sum_{i=2}^{n} w_{(i)}\right)^2 + \sum_{i=2}^{n} w_{(i)}^2 \right) \left(\sum_{j=1}^{n} x_j\right)} .$$

Then the sign of $D^*(\eta)$ is the same as the one for

$$\frac{x_s - x_t - \eta}{\sum_{j=1}^{n} x_j} + \frac{w_t - w_s}{\sum_{j=1}^{n} w_j} .$$

This completes the proof of the theorem. ▌

The next theorem shows that if all the users are given extra amounts of network resource proportionally according to their priority levels, then the fairness of the distribution will not decrease.

Theorem 6. For any $\delta > 0$,

$$G^*\left(x_1 + \frac{\delta w_1}{\sum_{j=1}^{n} w_j}, x_2 + \frac{\delta w_2}{\sum_{j=1}^{n} w_j}, \cdots, x_n + \frac{\delta w_n}{\sum_{j=1}^{n} w_j}; w_1, w_2, \cdots, w_n \right) \geq G^*\left(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n\right). \quad (10)$$

Proof. It can be shown that,

$$G^*\left(x_1 + \frac{\delta w_1}{\sum_{j=1}^{n} w_j}, x_2 + \frac{\delta w_2}{\sum_{j=1}^{n} w_j}, \cdots, x_n + \frac{\delta w_n}{\sum_{j=1}^{n} w_j}; w_1, w_2, \cdots, w_n \right)$$

$$=1-\frac{\left(\sum_{i=1}^{n} w_i\right)^2 \cdot \sum_{i=1}^{n}\left[\frac{x_i+\frac{\delta w_i}{\sum_{j=1}^{n} w_j}}{\sum_{j=1}^{n}\left(x_j+\frac{\delta w_j}{\sum_{k=1}^{n} w_k}\right)}-\frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2}{\left(\sum_{i=2}^{n} w_{(i)}\right)^2+\sum_{i=2}^{n} w_{(i)}^2}=1-\frac{\left(\sum_{i=1}^{n} w_i\right)^2 \cdot \sum_{i=1}^{n}\left[\frac{x_i+\frac{\delta w_i}{\sum_{j=1}^{n} w_j}}{\sum_{j=1}^{n} x_j+\delta}-\frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2}{\left(\sum_{i=2}^{n} w_{(i)}\right)^2+\sum_{i=2}^{n} w_{(i)}^2}.$$

Now it is desired to show that

$$\sum_{i=1}^{n}\left[\frac{x_i+\frac{\delta w_i}{\sum_{j=1}^{n} w_j}}{\sum_{j=1}^{n} x_j+\delta}-\frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2$$

is a decreasing function in δ. Actually,

$$\sum_{i=1}^{n}\left[\frac{x_i+\frac{\delta w_i}{\sum_{j=1}^{n} w_j}}{\sum_{j=1}^{n} x_j+\delta}-\frac{w_i}{\sum_{j=1}^{n} w_j}\right]^2=\sum_{i=1}^{n}\left[\frac{\left(x_i+\frac{\delta w_i}{\sum_{j=1}^{n} w_j}\right)-\frac{w_i}{\sum_{j=1}^{n} w_j}\left(\sum_{j=1}^{n} x_j+\delta\right)}{\sum_{j=1}^{n} x_j+\delta}\right]^2=\sum_{i=1}^{n}\left[\frac{x_i-\frac{w_i}{\sum_{j=1}^{n} w_j}\left(\sum_{j=1}^{n} x_j\right)}{\sum_{j=1}^{n} x_j+\delta}\right]^2.$$

This ends the proof. ▌

III. CONCLUSION AND DISCUSSION

After being proposed by Jain, Chiu, and Hawe [1] in 1984, the fairness score function $F(x_1, x_2, \cdots, x_n)$ described in (1) has been used to evaluate the fairness of the network resource distribution. Chen and Zhang [6] proposed another fairness score function $G(x_1, x_2, \cdots, x_n)$ which better fits the real world situation in some cases. The fairness function $G(x_1, x_2, \cdots, x_n)$, however, assumes that all the users are at the same priority level, i.e., all the users are supposed to be treated equally. In this paper, the situation that the users can have different priority levels is considered. A modified fairness score function $G^*(x_1, x_2, \cdots, x_n; w_1, w_2, \cdots, w_n)$ is proposed. The proposed fairness score function keeps all the nice properties of $G(x_1, x_2, \cdots, x_n)$.

Some authors recommend dividing the amounts that the users receive by their corresponding priority factors if the users are at different priority levels. More specifically, let $x_1, x_2, \cdots, x_n$ be the amounts of resource the users receive respectively, and let $w_1, w_2, \cdots, w_n$ be the corresponding priority factors of these users. Define

$$y_1 = \frac{x_1}{w_1}, y_2 = \frac{x_2}{w_2}, \cdots, y_n = \frac{x_n}{w_n}.$$

The fairness score function in (1) or (2) will then be used based on the quotients $y_1, y_2, \cdots, y_n$. It should be mentioned here that if this method is used, it may be misleading for the fairness evaluation. For instance, suppose there are only two users who are supposed to share the entire network resource. Suppose that their priority levels are 99 and 1, respectively. Consider the following two cases:

Case 1. Suppose user 1 (the user with the higher priority level) occupies the entire resource. Let $x_1 = x$ and $x_2 = 0$. This is, of course, not a perfectly fair distribution. However, if the huge difference between the two priority levels is considered, then the distribution is not bad at all. Actually, if the entire resource is perfectly fairly distributed to the two users according to their priority levels, then user 1 is supposed to use 99% of the resource while user 2 is supposed to use 1%. In this case, user 1 uses 100% of the resource. The difference between 99% and 100% is only 1%. Therefore, the fairness score should not change dramatically from the perfectly fair case to Case 1. However, if the fairness score function in (1) is used based on $y_1 = x/99$ and $y_2 = 0$, then the fairness score is 1/2. In other words, the distribution in this case is claimed to be completely unfair. If the fairness score function in (2) is used based on $y_1 = x/99$ and $y_2 = 0$, then the fairness score is 0. It should also be concluded that the distribution in this case is completely unfair. On the other hand, if the fairness score function in (3) is used, then the fairness score for this case is 9800/9801. This makes more sense in this application.

Case 2. Suppose user 2 (the user with the lower priority level) occupies the entire resource. Let $x_1 = 0$ and $x_2 = x$. This is a completely unfair distribution. If the fairness score function in (1) is used, then the fairness score is still 1/2. If the fairness score function in (2) is used, the fairness score is still 0. In other words, the fairness score functions in (1) and (2) cannot recognize the difference of the network resource distributions between the above two cases when the method of dividing the amounts by the corresponding priority levels is used. On the other hand, if the fairness score function in (3) is used, then the fairness score for Case 2 is 0. It means that this is the worst resource allocation.

Based on the above discussion, it can be concluded that the fairness score function defined in (3) performs better in evaluating the fairness of the network resource distribution when the users are at different priority levels.

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