# Correctness of Dijkstra's algorithm

- **Invariant**: When vertex u is deleted from the priority queue, d[u] is the correct length of the shortest path from the source s to vertex u.
  - Additionally, the value d[u] will never change after that iteration.

- **Proof**:
  - It is true for the first iteration. Why?
  - It is true for all subsequent iterations. Why?
    - Because you always pick the vertex with the smallest d[] value.

# Figure 14.38

Worst-case running times of various graph algorithms

Shortest Path between a pair of vertices

| TYPE OF GRAPH PROBLEM | RUNNING TIME | COMMENTS |
| --- | --- | --- |
| Unweighted | $O(\|E\|)$ | Breadth-first search |
| Weighted, no negative edges | $O(\|E\| \log \|V\|)$ | Dijkstra's algorithm |
| Weighted, negative edges | $O(\|E\| \cdot \|V\|)$ | Bellman–Ford algorithm |
| Weighted, acyclic | $O(\|E\|)$ | Uses topological sort |

# Polynomial-time computations

- An algorithm has time complexity $O(T(n))$ if it runs in time at most $cT(n)$ for <u>every</u> input of length $n$.
- An algorithm is a polynomial-time algorithm if its time complexity is $O(p(n))$, where $p(n)$ is polynomial in $n$.

# Polynomials

- If $f(n)$ = polynomial function in $n$,
  then $f(n) = O(n^c)$, for some fixed constant $c$
- If $f(n)$ = exponential (super-polynomial) function in $n$,
  then $f(n) = \omega(n^c)$, for any constant $c$
- Composition of polynomial functions are also polynomial, i.e.,
  $f(g(n))$ = polynomial if $f()$ and $g()$ are polynomial
- If an algorithm calls another polynomial-time subroutine a polynomial number of times, then the time complexity is polynomial.

# The class $\mathcal{P}$

- A problem is in $\mathcal{P}$ if there exists a polynomial-time algorithm that solves the problem.
- Examples of $\mathcal{P}$
  - ***DFS:*** Linear-time algorithm exists
  - ***Sorting:*** O(n log n)-time algorithm exists
  - ***Bubble Sort:*** Quadratic-time algorithm $O(n^2)$
  - ***APSP:*** Cubic-time algorithm $O(n^3)$
- $\mathcal{P}$ is therefore a class of problems (not algorithms)!

# The class $\mathcal{NP}$

- A problem is in $\mathcal{NP}$ if there exists a non-deterministic polynomial-time algorithm that solves the problem.
- A problem is in $\mathcal{NP}$ if there exists a (deterministic) polynomial-time algorithm that verifies a solution to the problem.
- All problems in $\mathcal{P}$ are in $\mathcal{NP}$

# TSP: Traveling Salesperson Problem

- **Input:**
  - Weighted graph, G
  - Length bound, B
- **Output:**
  - Is there a traveling salesperson tour in G of length at most B?
- Is TSP in $\mathcal{NP}$?
  - YES. Easy to verify a given solution.
- Is TSP in $\mathcal{P}$?
  - OPEN!
  - One of the greatest unsolved problems of this century!
  - Same as asking: **Is $\mathcal{P} = \mathcal{NP}$?**

# So, what is *NP - Complete*?

- *NP - Complete* problems are the “hardest” problems in *NP*.
- We need to formalize the notion of “hardest”.

# Terminology

- Problem:
  - An abstract problem is a function (relation) from a set I of instances of the problem to a set S of solutions.

    $p: I \rightarrow S$
  - An instance of a problem *p* is obtained by assigning values to the parameters of the abstract problem.
  - Thus, describing the set of all instances (I.e., possible inputs) and the set of corresponding outputs defines a problem.
- Algorithm:
  - An algorithm that solves problem *p* must give correct solutions to all instances of the problem.
- Polynomial-time algorithm:

# Terminology (Cont'd)

- Input Length:
  - length of an encoding of an instance of the problem.
  - Time and space complexities are written in terms of it.
- Worst-case time/space complexity of an algorithm
  - Is the maximum time/space required by the algorithm on any input of length n.
- Worst-case time/space complexity of a problem
  - UPPER BOUND: worst-case time complexity of best existing algorithm that solves the problem.
  - LOWER BOUND: (provable) worst-case time complexity of best algorithm (need not exist) that could solve the problem.
  - LOWER BOUND ≤ UPPER BOUND
- Complexity Class $\mathcal{P}$ :
  - Set of all problems *p* for which polynomial-time algorithms exist

# Terminology (Cont'd)

- Decision Problems:
  - These are problems for which the solution set is {yes, no}
  - Example: Does a given graph have an odd cycle?
  - Example: Does a given weighted graph have a TSP tour of length at most B?
- Complement of a decision problem:
  - These are problems for which the solution is "complemented".
  - Example: Does a given graph NOT have an odd cycle?
  - Example: Is every TSP tour of a given weighted graph of length greater than B?
- Optimization Problems:
  - These are problems where one is maximizing (or minimizing) some objective function.
  - Example: Given a weighted graph, find a MST.
  - Example: Given a weighted graph, find an optimal TSP tour.
- Verification Algorithms:
  - Given a problem instance i and a certificate s, is s a solution for instance i?

# Terminology (Cont'd)

- Complexity Class ***P*** :
  - Set of all problems *p* for which polynomial-time algorithms exist.
- Complexity Class ***NP*** :
  - Set of all problems *p* for which polynomial-time verification algorithms exist.
- Complexity Class ***co-NP*** :
  - Set of all problems *p* for which polynomial-time verification algorithms exist for their **complements**, i.e., their complements are in *NP.*

# Terminology (Cont'd)

- **Reductions:** $p_1 \rightarrow p_2$
  - A problem $p_1$ is reducible to $p_2$, if there exists an algorithm R that takes an instance $i_1$ of $p_1$ and outputs an instance $i_2$ of $p_2$, with the constraint that the solution for $i_1$ is YES if and only if the solution for $i_2$ is YES.
  - Thus, R converts YES (NO) instances of $p_1$ to YES (NO) instances of $p_2$.
- Polynomial-time reductions: $p_1 \xrightarrow{P} p_2$
  - Reductions that run in polynomial time.

- If $p_1 \xrightarrow{P} p_2$, then
  - If $p_2$ is easy, then so is $p_1$. $p_2 \in \mathcal{P} \Rightarrow p_1 \in \mathcal{P}$
  - If $p_1$ is hard, then so is $p_2$. $p_1 \notin \mathcal{P} \Rightarrow p_2 \notin \mathcal{P}$

# What are *NP-Complete* problems?

- These are the hardest problems in *NP*.
- A problem p is *NP-Complete* if
  - there is a polynomial-time reduction from **every** problem in *NP* to p.
  - p ∈ *NP*
- How to prove that a problem is *NP-Complete*?

- **Cook's Theorem**: [1972]
  - The SAT problem is *NP-Complete*.

**Steve Cook, Richard Karp, Leonid Levin**

# *NP-Complete* vs *NP-Hard*

- A problem p is *NP-Complete* if
  - there is a polynomial-time reduction from **every** problem in *NP* to p.
  - p $\in$ *NP*
- A problem p is *NP-Hard* if
  - there is a polynomial-time reduction from **every** problem in *NP* to p.

# The SAT Problem: an example

- Consider the boolean expression:

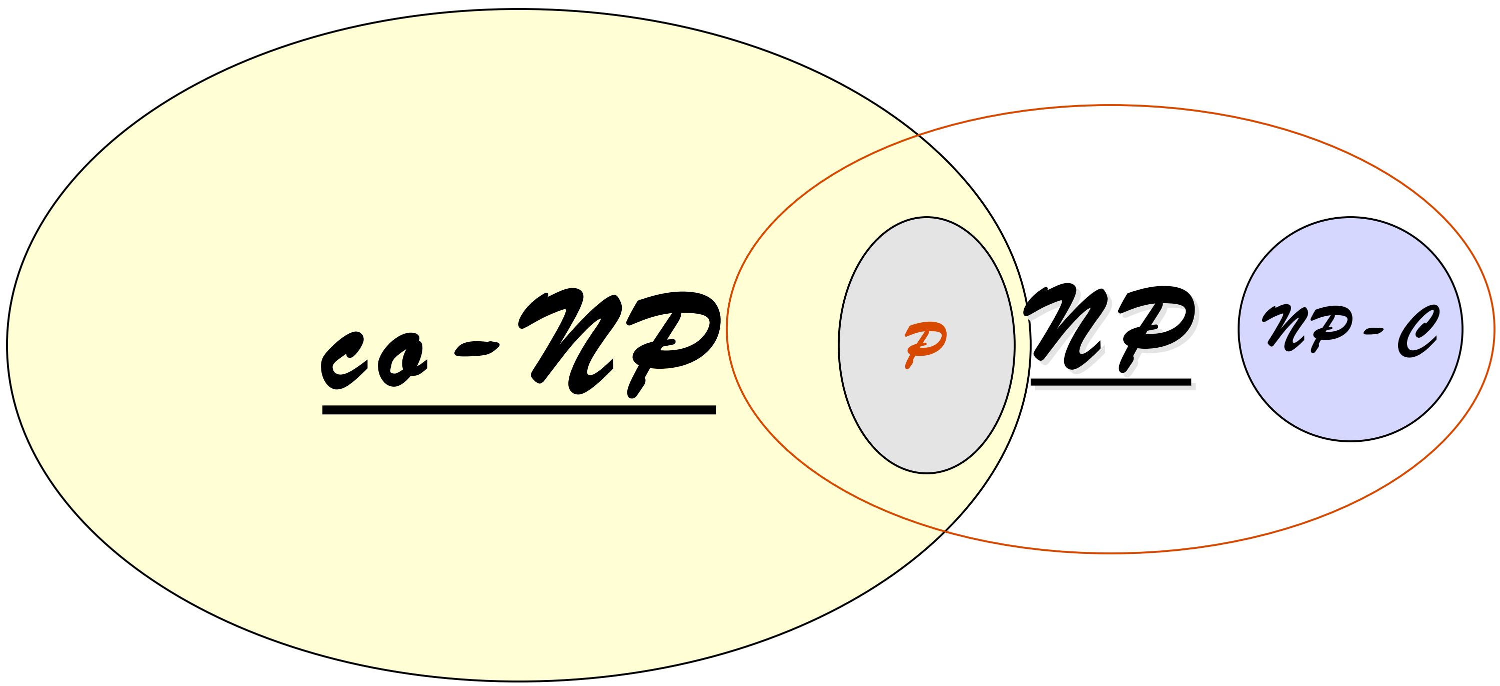
  $C = (a \vee \neg b \vee c) \wedge (\neg a \vee d \vee \neg e) \wedge (a \vee \neg d \vee \neg c)$
- Is $C$ satisfiable?
- Does there exist a True/False assignments to the boolean variables a, b, c, d, e, such that $C$ is True?
- Set a = True and d = True. The others can be set arbitrarily, and $C$ will be true.
- If $C$ has 40,000 variables and 4 million clauses, then it becomes hard to test this.
- If there are n boolean variables, then there are $2^n$ different truth value assignments.
- However, a solution can be quickly verified!

# The SAT (Satisfiability) Problem

- Input: Boolean expression $C$ in Conjunctive normal form (CNF) in $n$ variables and $m$ clauses.
- Question: Is $C$ satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = \left(y_1^i \vee y_2^i \vee \cdots \vee y_{k_i}^i\right)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression $C$ that makes it true.
- Steve Cook showed that the problem of deciding whether a non-deterministic Turing machine $T$ accepts an input $w$ or not can be written as a boolean expression $C_T$ for a SAT problem. The boolean expression will have length bounded by a polynomial in the size of $T$ and $w$.

> • How to now prove Cook's theorem? Is SAT in $\mathcal{NP}$?
>
> • Can every problem in $\mathcal{NP}$ be poly. reduced to it ?

# The problem classes and their relationships



# More *NP-Complete* problems

## 3SAT

- Input: Boolean expression $C$ in Conjunctive normal form (CNF) in $n$ variables and $m$ clauses. Each clause has at most three literals.
- Question: Is $C$ satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = \left(y_1^i \vee y_2^i \vee y_3^i\right)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression $C$ that makes it true.

3SAT is *NP-Complete.*

# More *NP-Complete* problems?

## 2SAT

- Input: Boolean expression $C$ in Conjunctive normal form (CNF) in $n$ variables and $m$ clauses. Each clause has at most three literals.
- Question: Is $C$ satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ \dots \ \wedge C_m$
  - Where each $C_i = (y_1^i \vee y_2^i)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression $C$ that makes it true.

2SAT is in *P*.

# 3SAT is *NP-Complete*

- 3SAT is in *NP*.
- SAT can be reduced in polynomial time to 3SAT.
- This implies that every problem in *NP* can be reduced in polynomial time to 3SAT. Therefore, 3SAT is *NP-Complete*.
- So, we have to design an algorithm such that:
- Input: an instance *C* of SAT
- Output: an instance *C'* of 3SAT such that satisfiability is retained. In other words, *C* is satisfiable if and only if *C'* is satisfiable.

# 3SAT is *NP-Complete*

- Let $C$ be an instance of SAT with clauses $C_1, C_2, \ldots, C_m$
- Let $C_i$ be a disjunction of $k > 3$ literals.

  $C_i = \quad y_1 \vee y_2 \vee \ldots \vee y_k$
- Rewrite $C_i$ as follows:

$$
\begin{aligned}
C'_i = \quad & (y_1 \vee y_2 \vee z_1) \wedge \\
& (\neg z_1 \vee y_3 \vee z_2) \wedge \\
& (\neg z_2 \vee y_4 \vee z_3) \wedge \\
& \ldots \\
& (\neg z_{k-3} \vee y_{k-1} \vee y_k)
\end{aligned}
$$

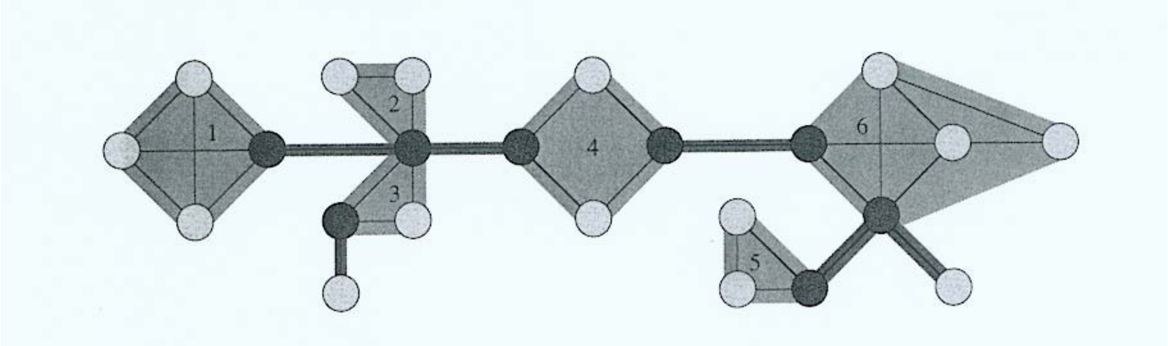
- Claim: $C_i$ is satisfiable if and only if $C'_i$ is satisfiable.

# 2SAT is in $\mathcal{P}$

- If there is only one literal in a clause, it must be set to true.
- If there are two literals in some clause, and if one of them is set to false, then the other must be set to true.
- Using these constraints, it is possible to check if there is some inconsistency.
- How? Homework problem!

# The CLIQUE Problem

- A **clique** is a completely connected subgraph.

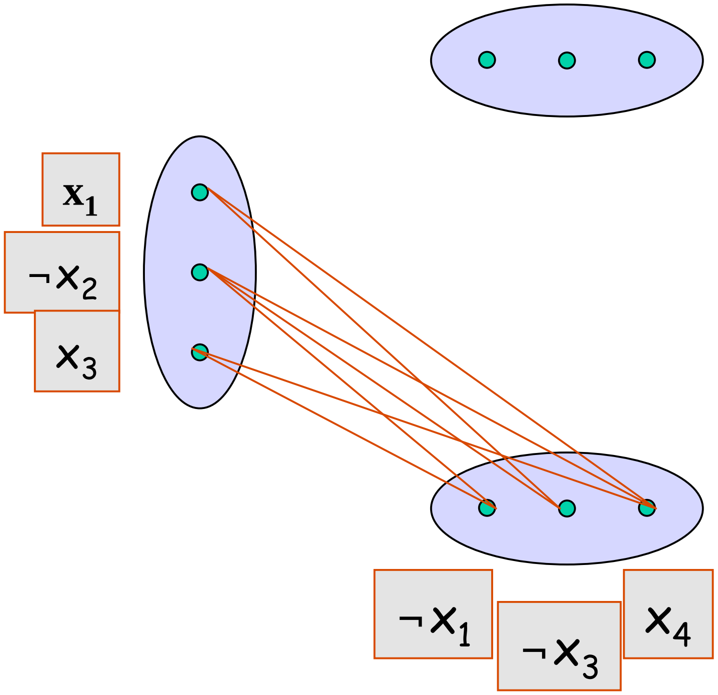



CLIQUE

- Input: Graph G(V,E) and integer k
- Question: Does G have a clique of size k?

# CLIQUE is *NP-Complete*

- CLIQUE is in *NP*.
- Reduce 3SAT to CLIQUE in polynomial time.
- $F = (x_1 \vee \neg x_2 \vee x_3)\ (\neg x_1 \vee \neg x_3 \vee x_4)\ (x_2 \vee x_3 \vee \neg x_4)\ (\neg x_1 \vee \neg x_2 \vee x_3)$

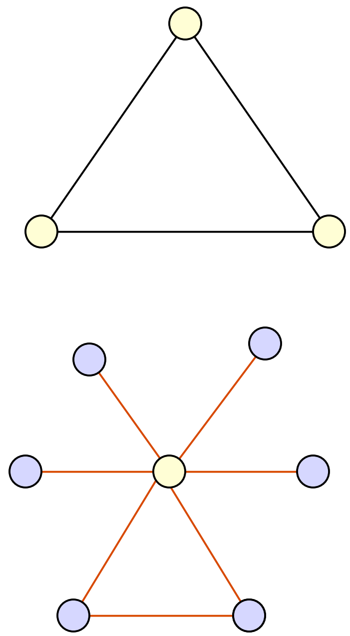



F is satisfiable if and only if G has a clique of size k where k is the number of clauses in F.

# Vertex Cover

A vertex cover is a set of vertices that "covers" all the edges of the graph.

Examples

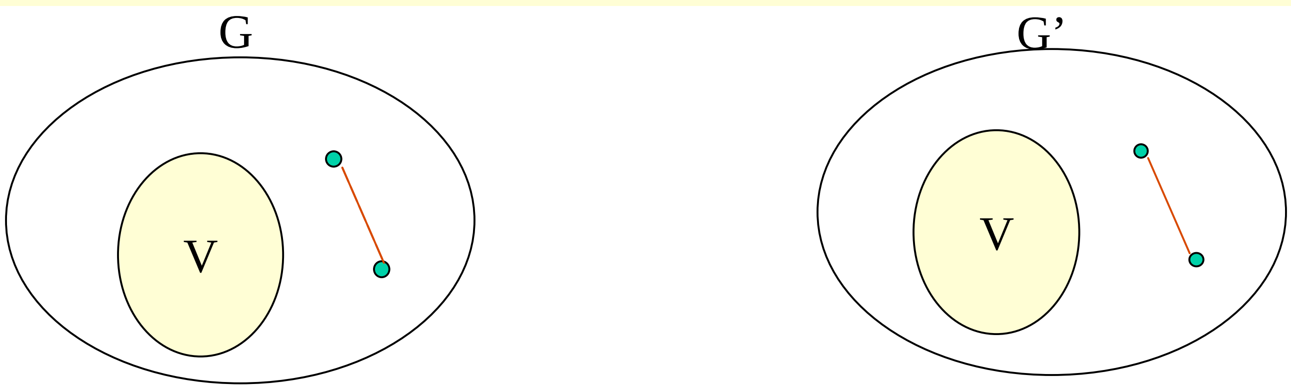


# Vertex Cover (VC)

Input: Graph G, integer k

Question: Does G contain a vertex cover of size k?

- VC is in *NP*.
- polynomial-time reduction from CLIQUE to VC.
- Thus VC is *NP-Complete.*




Claim: G' has a clique of size k' if and only if G has a VC of size k = n – k'

# Hamiltonian Cycle Problem (HCP)

Input: Graph G

Question: Does G contain a hamiltonian cycle?

- HCP is in *NP*.
- There exists a polynomial-time reduction from 3SAT to HCP.
- Thus HCP is *NP-Complete.*

- Notes/animations by Yi Ge!