COT 5407: Introduction to Algorithms

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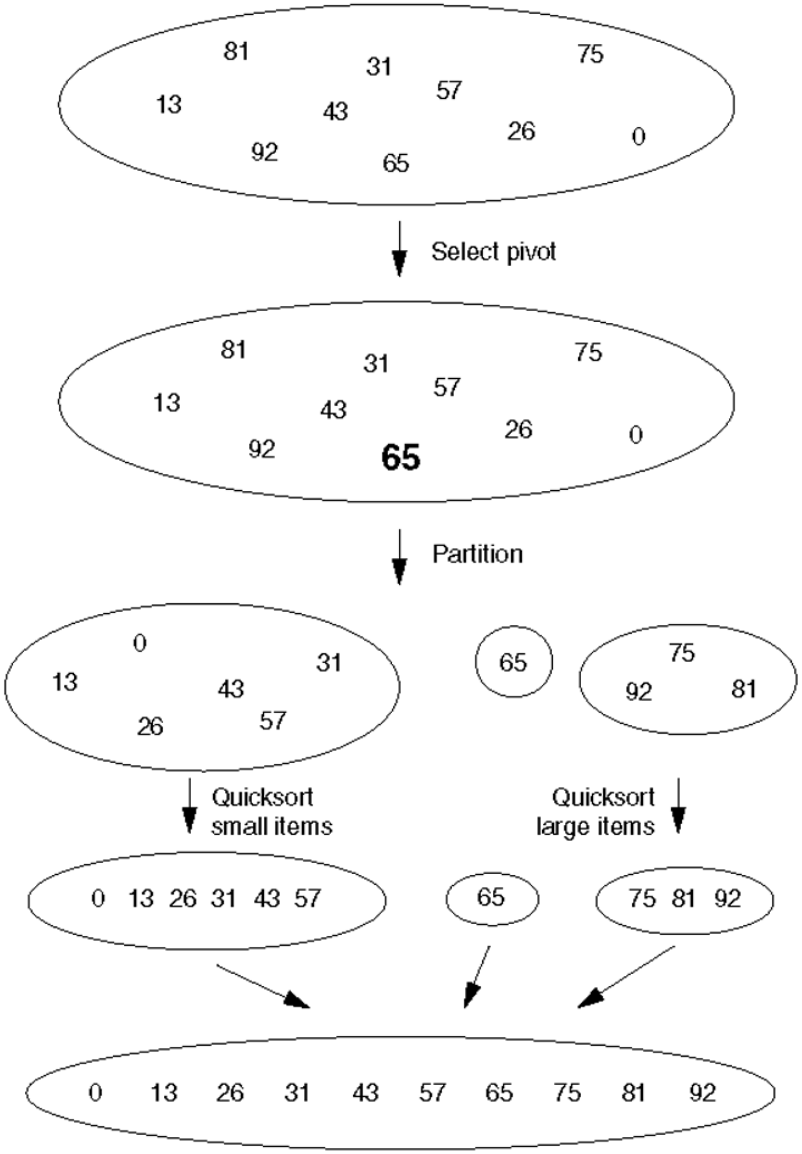
http://www.cis.fiu.edu/~giri/teach/5407S17.html

https://moodle.cis.fiu.edu/v3.1/course/view.php?id=1494

Figure 8.10 Quicksort



Data Structures & Problem Solving using JAVA/2E Mark Allen Weiss

Partition Algorithm

- Pick a pivot
- Compare each item to a pivot and create two lists:
  - L = list of all items smaller than the pivot
  - R = list of all items larger than the pivot

O(N) time

- One scan through the list is enough, but seems to need extra space
- How to design an **in-place partition algorithm**!

QuickSort

```
QUICKSORT($array\ A, int\ p, int\ r$)
if ($p < r$)
    then $q \leftarrow$ PARTITION($A, p, r$)
         QUICKSORT($A, p, q - 1$)
         QUICKSORT($A, q + 1, r$)
```

To sort array call QUICKSORT(A, 1, *length*[A]).

```
PARTITION($array\ A, int\ p, int\ r$)
$x \leftarrow A[r]$                    ▷ Choose pivot
$i \leftarrow p - 1$
for $j \leftarrow p$ to $r - 1$
    do if ($A[j] \leq x$)
        then $i \leftarrow i + 1$
             exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$
```

Page 146, CLRS

Time Complexity

Recurrence Relaton

- $T(N) = O(N) + T(N_1) + T(N_2)$

Average-Case Time Complexity

- On the average, $N_1 = N_2 = N/2$
- $T(N) = O(N) + 2T(N/2)$
- Thus, average-case complexity = $O(N \log N)$

Worst-Case Time Complexity

- Worst-case: Either N_1 or $N_2 = 0$
  - Thus, $T(N) = O(N) + T(N - 1)$
  - $T(N) = O(N^2)$

Variants of QuickSort

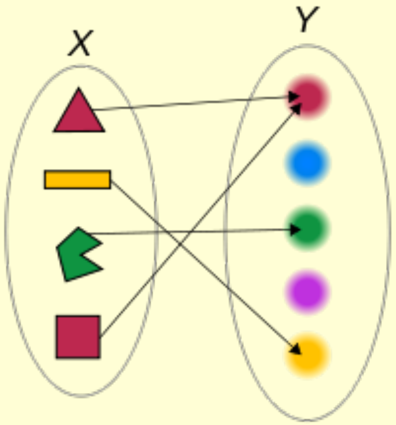
- Choice of Pivot
  - Random choice
  - Median of 3
  - Median
- Avoiding recursion on small subarrays
  - Invoking InsertionSort for small arrays

Sorting Algorithms

- SelectionSort
- InsertionSort
- BubbleSort
- ShakerSort
- MergeSort
- HeapSort
- QuickSort
- Bucket & Radix Sort
- Counting Sort

Definitions

Abstract Problem: defines a function from any allowable input to a corresponding output



Instance of a Problem: a specific input to abstract problem

Algorithm: well-defined computational procedure that takes an instance of a problem as input and produces the correct output

An Algorithm must <u>halt</u> on every input with <u>correct</u> output.

Algorithm Analysis

- Worst-case time complexity*
  - Worst possible time of all input instances of length N
- (Worst-case) space complexity
  - Worst possible spaceof all input instances of length N
- Average-case time complexity
  - Average time of all input instances of length N

Upper and Lower Bounds

- Time Complexity of a Problem
  - **Difficulty**: Since there can be many algorithms that solve a problem, what time complexity should we pick?
  - **Solution**: Define upper bounds and lower bounds within which the time complexity lies.
- What is the upper bound on time complexity of sorting?
  - ***Answer***: Since SelectionSort runs in worst-case $O(N^2)$ and MergeSort runs in $O(N \log N)$, either one works as an upper bound.
  - **Critical Point**: Among all upper bounds, the best is the lowest possible upper bound, i.e., time complexity of the best algorithm.
- What is the lower bound on time complexity of sorting?
  - **Difficulty**: If we claim that lower bound is $O(f(N))$, then we have to prove that no algorithm that sorts N items can run in worst-case time $o(f(N))$.

Lower Bounds

- Surprisingly, it is possible to prove lower bounds for many comparison-based problems.
- For any comparison-based problem, for any input of length N, if there are $P(N)$ possible solutions, then any algorithm must need $\boxed{\log_2(P(N))}$ to solve the problem.
- Binary Search on a list of N items has at least N + 1 possible solutions. Hence lower bound is
  - $\log_2(N+1)$.
- Sorting a list of N items has at least N! possible solutions. Hence lower bound is
  - $\log_2(N!) = O(N \log N)$
- Thus, MergeSort is an optimal algorithm.
  - Because its worst-case time complexity equals lower bound!

Beating the Lower Bound

- Bucket Sort
  - Runs in time O(N+K) given N integers in range [a+1, a+K]
  - If K = O(N), we are able to sort in O(N)
  - How is it possible to beat the lower bound?
  - Only because we know more about the data.
  - If nothing is know about the data, the lower bound holds.
- Radix Sort
  - Runs in time O(d(N+K)) given N items with d digits each in range [1,K]
- Counting Sort
  - Runs in time O(N+K) given N items in range [a+1, a+K]

Bucket Sort

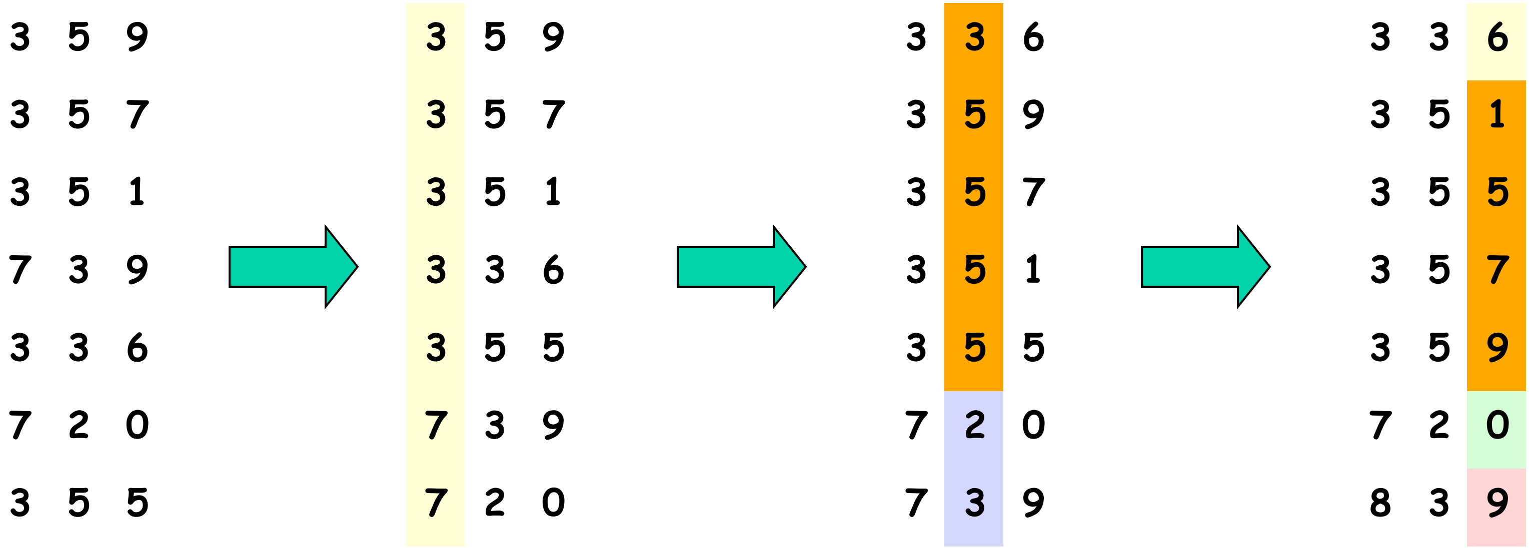
- N **integer** values in the range [a..a+m-1]
- For e.g., sort a list of 50 scores in the range [0..9].
- Algorithm
  - Make m buckets [a..a+m-1]
  - As you read elements throw into appropriate bucket
  - Output contents of buckets [0..m] in that order
- Time O(N+m)

- **Warning**: This algorithm cannot be used for "infinite-precision" real numbers, even if the range of values is specified.

Stable Sort

- A sort is stable if equal elements appear in the same order in both the input and the output.
- Which sorts are stable? Homework!

Radix Sort



3 5 9	3 5 9	3 3 6	3 3 6
3 5 7	3 5 7	3 5 9	3 5 1
3 5 1	3 5 1	3 5 7	3 5 5
7 3 9	3 3 6	3 5 1	3 5 7
3 3 6	3 5 5	3 5 5	3 5 9
7 2 0	7 3 9	7 2 0	7 2 0
3 5 5	7 2 0	7 3 9	8 3 9

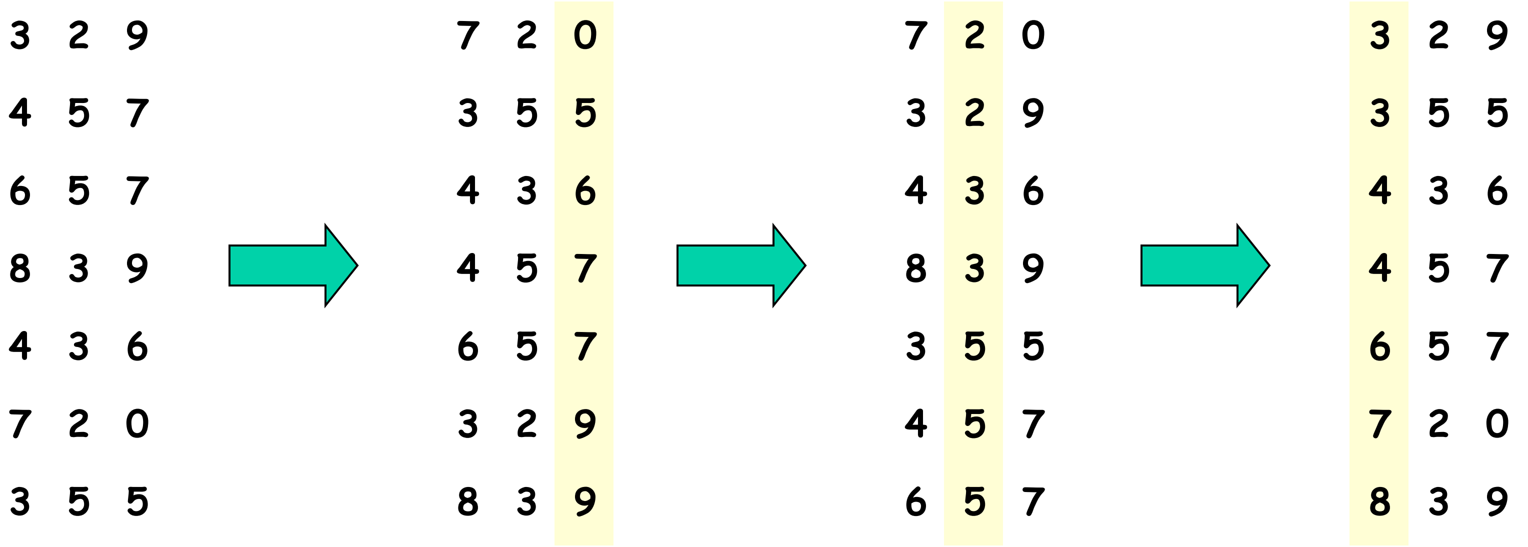
Algorithm

```
for i = 1 to d do
        sort array A on digit i using any sorting algorithm
```

Time Complexity: $O((N+m) + (N+m^2) + \ldots + (N+m^d))$

Space Complexity: $O(m^d)$

Radix Sort



3 2 9	7 2 0	7 2 0	3 2 9
4 5 7	3 5 5	3 2 9	3 5 5
6 5 7	4 3 6	4 3 6	4 3 6
8 3 9	4 5 7	8 3 9	4 5 7
4 3 6	6 5 7	3 5 5	6 5 7
7 2 0	3 2 9	4 5 7	7 2 0
3 5 5	8 3 9	6 5 7	8 3 9

Algorithm

```
for i = 1 to d do
        sort array A on digit i using a stable sort algorithm
```

Time Complexity: $O((n+m)d)$

- **Warning**: This algorithm cannot be used for "infinite-precision" real numbers, even if the range of values is specified.

Counting Sort

Initial Array

1	2	3	4	5	6	7	8
2	5	3	0	2	3	0	3

Counts

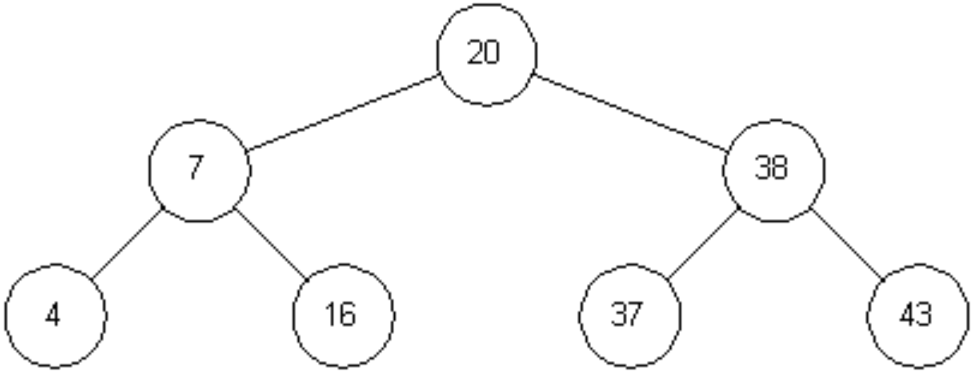
0	1	2	3	4	5
2	0	2	3	0	1

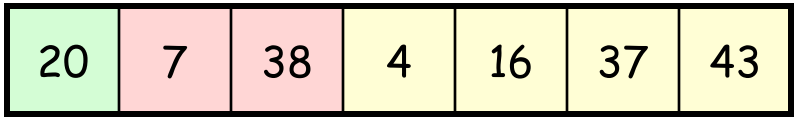
Cumulative Counts

0	1	2	3	4	5
2	2	4	7	7	8

- **Warning**: This algorithm cannot be used for “infinite-precision” real numbers, even if the range of values is specified.

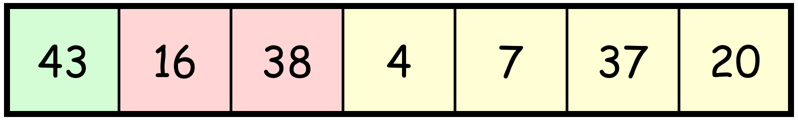
Storing binary trees as arrays



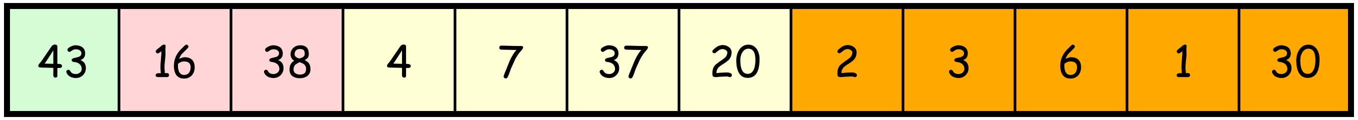


20	7	38	4	16	37	43

Heaps (Max-Heap)



43	16	38	4	7	37	20



43	16	38	4	7	37	20	2	3	6	1	30

HEAP represents a binary tree stored as an array such that:

- Tree is filled on all levels except last
- Last level is filled from left to right
- Left & right child of i are in locations 2i and 2i+1
- HEAP PROPERTY:
  Parent value is at least as large as child’s value

HeapSort

- First convert array into a heap (BUILD-MAX-HEAP, p133)
- Then convert heap into sorted array (HEAPSORT, p136)

Animation Demos

http://www-cse.uta.edu/~holder/courses/cse2320/lectures/applets/sort1/heapsort.html

http://cg.scs.carleton.ca/~morin/misc/sortalg/

HeapSort: Part 1

```
MAX-HEAPIFY($array\ A, int\ i$)
    ▷ Assume subtree rooted at $i$ is not a heap;
    ▷ but subtrees rooted at children of $i$ are heaps
    $l \leftarrow$ LEFT[$i$]
    $r \leftarrow$ RIGHT[$i$]
    if (($l \leq heap\text{-}size[A]$) and ($A[l] > A[i]$))
        then $largest \leftarrow l$
        else $largest \leftarrow i$
    if (($r \leq heap\text{-}size[A]$) and ($A[r] > A[largest]$))
        then $largest \leftarrow r$
    if ($largest \neq i$)
        then exchange $A[i] \leftrightarrow A[largest]$
             MAX-HEAPIFY($A, largest$)
```

O(height of node in location i) = O(log(size of subtree))

p154, CLRS

HeapSort: Part 2

```
BUILD-MAX-HEAP($array\ A$)
    $heap\text{-}size[A] \leftarrow length[A]$
    for $i \leftarrow \lfloor length[A]/2 \rfloor$ downto 1
        do MAX-HEAPIFY($A, i$)
```

HeapSort: Part 2

```
BUILD-MAX-HEAP(array A)
    heap-size[A] ← length[A]
    for i ← $\lfloor length[A]/2 \rfloor$ downto 1
        do MAX-HEAPIFY(A, i)
```

```
HEAPSORT(array A)
    BUILD-MAX-HEAP(A)
    for i ← length[A] downto 2
        do exchange A[1] ↔ A[i]
            heap-size[A] ← heap-size[A] − 1
            MAX-HEAPIFY(A, 1)
```

O(log n)

Total:
O(nlog n)

Build-Max-Heap Analysis

We need to compute:

$$\boxed{\sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^n}}$$

$$\text{We know that } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\text{Differnetiating both sides, we get } \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$\text{Multiplying both sides by } x, \text{weget } \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

$$\text{Setting } x = 1/2, \text{we can show that } \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^n} \leq 2$$