COT 5407: Introduction to Algorithms

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http://www.cis.fiu.edu/~giri/teach/5407S17.html

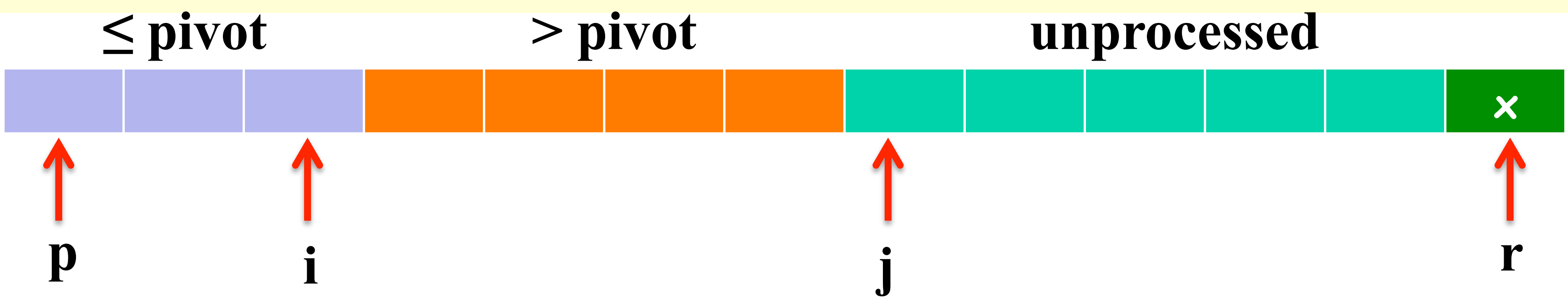
https://moodle.cis.fiu.edu/v3.1/course/view.php?id=1494

Homework

- **Read Guidelines and Follow Instructions!**
- **Statement of Collaboration**
  - Take it **seriously.**
  - If true, **reproduce** the statement faithfully.
  - For each problem, explain **separately** the sources and your collaborations with other people.
  - Your homework **will not be graded** without the statement.
- **Extra Credit Problem**
  - You can turn it in any time within a month or until last class day, whichever is earlier.
  - If you are not sure of your solution, don't waste my time.
  - You will NOT get partial credit on an extra credit problem.
  - Submit it separately and label it appropriately.

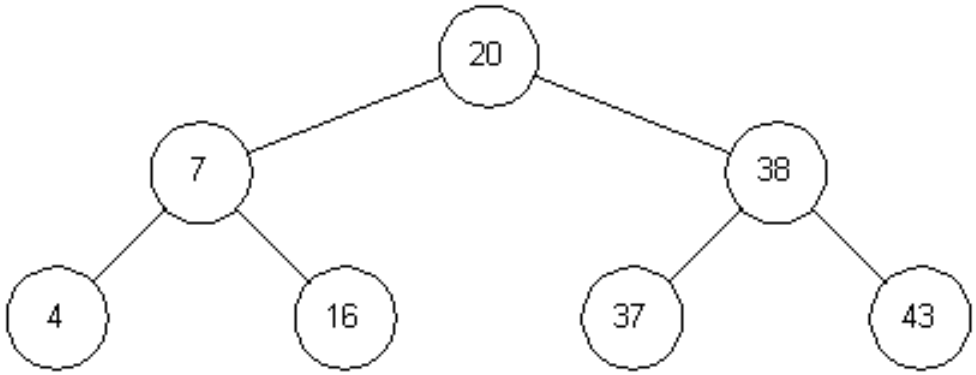
QuickSort: variant for Partition

- At the start of iteration j,
  - A[1..i] has elements that are smaller than or equal to pivot x
  - A[i+1..j-1] has elements that are larger than pivot x
  - A[j..r-1] have not yet been processed
  - A[r] has the pivot x
- Try to prove this invariant!



•**Warning:** Quicksort cannot be used if a sorting algorithm is needed that runs in time O(n log n) in the worst case.

Storing binary trees as arrays



20	7	38	4	16	37	43

Heaps (Max-Heap)

43	16	38	4	7	37	20

43	16	38	4	7	37	20	2	3	6	1	30

HEAP represents a binary tree stored as an array such that:

- Tree is filled on all levels except the last level
- Last level is filled from left to right
- Left & right child of i are in locations $2i$ and $2i+1$
- HEAP PROPERTY:

  Parent value is at least as large as child's value

HeapSort

- First convert array into a heap (BUILD-MAX-HEAP, p157)
- Then convert heap into sorted array (HEAPSORT, p160)

Animation Demos

http://www-cse.uta.edu/~holder/courses/cse2320/lectures/applets/sort1/heapsort.html

http://cg.scs.carleton.ca/~morin/misc/sortalg/

HeapSort: Part 1

```
MAX-HEAPIFY(array A, int i)
    ▷ Assume subtree rooted at i is not a heap;
    ▷ but subtrees rooted at children of i are heaps
l ← LEFT[i]
r ← RIGHT[i]
if ((l ≤ heap-size[A]) and (A[l] > A[i]))
    then largest ← l
    else  largest ← i
if ((r ≤ heap-size[A]) and (A[r] > A[largest]))
    then largest ← r
if (largest ≠ i)
    then exchange A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

p154, CLRS

Analysis of Max-Heapify

```
MAX-HEAPIFY(array A, int i)
   ▷ Assume subtree rooted at i is not a heap;
   ▷ but subtrees rooted at children of i are heaps
 l ← LEFT[i]
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 if ((l ≤ heap-size[A]) and (A[l] > A[i]))
     then largest ← l
     else  largest ← i
 if ((r ≤ heap-size[A]) and (A[r] > A[largest]))
     then largest ← r
 if (largest ≠ i)
     then exchange A[i] ↔ A[largest]
          MAX-HEAPIFY(A, largest)
```

- $T(N) \leq T(2N/3) + O(1)$
- When called on node i, either it terminates with O(1) steps or makes a recursive call on node at lower level
- At most 1 call per level
- Time Complexity = O(level of node i) = $O(h_i)$ = O(log N)

HeapSort: Part 2

BUILD-MAX-HEAP(*array* A)

$heap\text{-}size[A] \leftarrow length[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
        do MAX-HEAPIFY(A, i)

HeapSort: Part 2

BUILD-MAX-HEAP(*array* A)

$heap\text{-}size[A] \leftarrow length[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
    do MAX-HEAPIFY(A, i)

HEAPSORT(*array* A)

BUILD-MAX-HEAP(A)
for $i \leftarrow length[A]$ **downto** 2
    do exchange $A[1] \leftrightarrow A[i]$
        $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
        MAX-HEAPIFY($A, 1$)

O(log n)

Total: O(nlog n)

HeapSort: Part 2

```
BUILD-MAX-HEAP(array A)
heap-size[A] ← length[A]
for i ← ⌊length[A]/2⌋ downto 1
        do MAX-HEAPIFY(A, i)
```

- For n/2 nodes, height is 1 and # of comparisons = 0,
- For n/4 nodes, height is 2 and # of comparisons = 1,
- For n/8 nodes, height is 3 and # of comparisons = 2, …
- Total = summation ((height -1) X # of nodes at that height)
- Total = summation ((height – 1) X $N/2^{height}$)
- Total ≤ summation (height X $N/2^{height}$)
- Total ≤ N X summation (height X $1/2^{height}$)

Build-Max-Heap Analysis

We need to compute: $$\sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}$$

Build-Max-Heap: O(N)

$$\text{We know that } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\text{Differnetiating both sides, we get } \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$\text{Multiplying both sides by } x, \text{weget } \sum_{k=0}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

$$\text{Setting } x = 1/2, \text{we can show that } \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h} \leq 2$$

HeapSort

```
BUILD-MAX-HEAP(array A)
heap-size[A] ← length[A]
for i ← ⌊length[A]/2⌋ downto 1
        do MAX-HEAPIFY(A, i)
```

```
HEAPSORT(array A)
BUILD-MAX-HEAP(A)
for i ← length[A] downto 2
        do exchange A[1] ↔ A[i]
            heap-size[A] ← heap-size[A] − 1
            MAX-HEAPIFY(A, 1)
```

- Single call to Max-Heapify runs in O(h) time
- However, Build-Max-Heap runs in O(N) time
- HeapSort runs in O(N log N) time

Sorting Algorithms

- SelectionSort
- InsertionSort
- BubbleSort
- ShakerSort
- MergeSort
- HeapSort
- QuickSort
- Bucket & Radix Sort
- Counting Sort

Upper and Lower Bounds

- Time Complexity of a Problem
  - **Difficulty**: Since there can be many algorithms that solve a problem, what time complexity should we pick?
  - **Solution**: Define upper bounds and lower bounds within which the time complexity lies.
- What is the upper bound on time complexity of sorting?
  - **Answer**: Since SelectionSort runs in worst-case $O(N^2)$ and MergeSort runs in $O(N \log N)$, either one works as an upper bound.
  - **Critical Point**: Among all upper bounds, the best is the lowest possible upper bound, i.e., time complexity of the best algorithm.
- What is the lower bound on time complexity of sorting?
  - **Difficulty**: If we claim that lower bound is $O(f(N))$, then we have to prove that no algorithm that sorts N items can run in worst-case time $o(f(N))$.

Lower Bounds

- Surprisingly, it is possible to prove lower bounds for many comparison-based problems.
- For any comparison-based problem, for any input of length N, if there are $P(N)$ possible solutions, then any algorithm must need $\log_2(P(N))$ to solve the problem.
- Binary Search on a list of N items has at least N + 1 possible solutions. Hence lower bound is
  - $\log_2(N+1)$.
- Sorting a list of N items has at least N! possible solutions. Hence lower bound is
  - $\log_2(N!) = O(N \log N)$
- Thus, MergeSort is an optimal algorithm.
  - Because its worst-case time complexity equals lower bound!

Beating the Lower Bound

- Bucket Sort
  - Runs in time O(N+K) given N integers in range [a+1, a+K]
  - If K = O(N), we are able to sort in O(N)
  - How is it possible to beat the lower bound?
  - Only because we know more about the data.
  - If nothing is know about the data, the lower bound holds.
- Radix Sort
  - Runs in time O(d(N+K)) given N items with d digits each in range [1,K]
- Counting Sort
  - Runs in time O(N+K) given N items in range [a+1, a+K]

Bucket Sort

- N **integer** values in the range [a..a+m-1]
- For e.g., sort a list of 50 scores in the range [0..9].
- Algorithm
  - Make m buckets [a..a+m-1]
  - As you read elements throw into appropriate bucket
  - Output contents of buckets [0..m] in that order
- Time O(N+m)

- **Warning**: This algorithm cannot be used for "infinite-precision" real numbers, even if the range of values is specified.

Stable Sort

- A sort is stable if equal elements appear in the same order in both the input and the output.
- Which sorts are stable?

Radix Sort

3 5 9	3 5 9	3 3 6	3 3 6
3 5 7	3 5 7	3 5 9	3 5 1
3 5 1	3 5 1	3 5 7	3 5 5
7 3 9	3 3 6	3 5 1	3 5 7
3 3 6	3 5 5	3 5 5	3 5 9
7 2 0	7 3 9	7 2 0	7 2 0
3 5 5	7 2 0	7 3 9	8 3 9

Algorithm

for i = 1 **to** d **do**

sort array A on digit i using any sorting algorithm

Time Complexity: $O((N+m) + (N+m^2) + \ldots + (N+m^d))$

Space Complexity: $O(m^d)$

Radix Sort

3 2 9	7 2 0	7 2 0	3 2 9
4 5 7	3 5 5	3 2 9	3 5 5
6 5 7	4 3 6	4 3 6	4 3 6
8 3 9	4 5 7	8 3 9	4 5 7
4 3 6	6 5 7	3 5 5	6 5 7
7 2 0	3 2 9	4 5 7	7 2 0
3 5 5	8 3 9	6 5 7	8 3 9

Algorithm

for i = 1 **to** d **do**

sort array A on digit i using a stable sort algorithm

Time Complexity: O((n+m)d)

• **Warning**: This algorithm cannot be used for "infinite-precision" real numbers, even if the range of values is specified.

Counting Sort

Initial Array

1	2	3	4	5	6	7	8
2	5	3	0	2	3	0	3

Counts

0	1	2	3	4	5
2	0	2	3	0	1

Cumulative Counts

0	1	2	3	4	5
2	2	4	7	7	8

• **Warning**: This algorithm cannot be used for "infinite-precision" real numbers, even if the range of values is specified.