Panther ID (No name, please):

COP 5407: Intro. to Algorithms MIDTERM EXAM 2; Spring 2019

Max Score = 75 pts; Estimated Time = 75 minutes; Max Time = 6 hours

Read This: This is a "open book" exam and has a total of 4 pages. When tackling an algorithmic problem, provide a "Basic Idea" behind your solution first. If you are writing pseudocode, provide some comments for key portions or for "tricky" code. Make it easier for a reader to understand your pseudocode. Analyze the worst-case time complexity of all algorithms you are asked to design. Always attempt to argue why your algorithm is correct.

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1. [37] Design an augmented RB-tree, T, to store a set of items from a warehouse and to efficiently respond to a sequence of queries. Each warehouse item has a unique *key* value. The queries are from the following set:

   - ITEM-INSERT(T, k) – insert item with *key* k in tree T.
   - ITEM-DELETE(T, k) – delete item with *key* k from tree T.
   - COUNTITEMS(r, x, y) – return the number of the items with *key* in the range $[x, y]$ in the subtree rooted at node r.

   Clearly, ITEM-INSERT and ITEM-DELETE can be handled to run in $O(\log n)$ time by using a simple RB-tree (without augmentation). However, the COUNTITEMS queries may require you to augment the tree nodes with additional information. Without any augmented information stored in each node, one way to respond to the COUNTITEMS queries is to do a naive tree traversal that visits each node in the subtree rooted at r and does the necessary count. Even if you avoid visiting each branch and node of the tree, COUNTITEMS(r, x, y) could take as much as $O(n)$ time in a tree without augmentations. The following series of questions will help you to design the necessary augmented RB-tree data structure that will make COUNTITEMS more efficient in the worst case.

   (a) [5] State what **augmented** piece(s) of information you will store in each node (besides the standard *key, left, right, color*) in order to facilitate the above queries.

   (b) [3] Explain briefly why this information can be maintained in the presence of inserts and deletes in the RB-tree.

   (c) [25] Write down an efficient algorithm for COUNTITEMS(r, x, y).

   (d) [4] Analyze the worst-case time complexity of COUNTITEMS and show that it is **sublinear**.

2. [38] Carina plans to sail from Miami to Atlantis, which is at the diagonally opposite end of the Bermuda Square. To facilitate her planning, Carina has drawn herself a grid and placed Miami at coordinates $(0, 0)$ and Atlantis at coordinates (P, Q) in the northeast direction. Along the way, Carina could stop at various islands where she could sell the VR headsets that she produces in Miami. For each island, she knows how many headsets she can sell. However, the winds will only allow her to sail in the northeasterly direction. This means that if she decides to sail from a point at coordinates (x_1, y_1) to a point at coordinates (x_2, y_2), then $x_1 < x_2$ and $y_1 < y_2$. Help Carina to maximize her sale of VR headsets on her sail from Miami to Atlantis. The input to the problem includes the coordinates of Atlantis and the coordinates of the n islands along with the number of headsets that she can sell at each of those islands.

   The following questions are optional, except for (e), (g), and (i). They are meant to help you to solve this DP problem. Note that your answers to the optional question may help your thinking and fetch you partial credit.

   (a) Define notation for the inputs to the problem and for the desired output.

   (b) Write down a preliminary recursive solution for this problem.

   (c) Identify the list of subproblems to solve.

   (d) Describe the data structure to use for the memoization.

   (e) **You must solve this:** Write down a recurrence relation for solving the subproblems. Remember to provide clear and precise definitions of any new notation used in your recurrence.

   (f) Identify a hierarchy/ordering for solving the subproblems.

   (g) **You must solve this:** Put it all together to design a DP algorithm to output the maximum number of headsets Carina can sell on her sail from Miami to Atlantis.

   (h) Argue for the correctness of your algorithm.

   (i) **You must solve this:** Analyze the time complexity of the algorithm

   (j) [Xtra Credit] Modify your algorithm to output the islands that Carina should visit on her optimal sail route.

   (k) [Xtra Credit] Prove that a *greedy algorithm* cannot be designed for this problem, or design an efficient greedy algorithm.