Tree Augmentation

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The Problem: CodeChef CHN15E

• Given tree $T$, the **augmented tree** $G_T$ is defined as the graph obtained by joining every pair of vertices at distance 2 from each other.

• The problem is to construct $T$, given $G_T$. 
Simple Properties

• Vertices of $T$ and $G_T$ are the same.
• Let neighbors of vertex $v$ in $T$ be the set $N(v)$
• The set $\{v\} \cup N(v)$ forms a **clique** in $G_T$.
  - A subset of vertices in a graph forms a **clique** if all of them are connected by edges (i.e., no pair of vertices in this subset are missing an edge)
• A **maximal clique** is a set of vertices that forms a clique for which no superset is a clique.
More Properties

• For a tree T with n vertices, the augmented tree $G_T$ has at most n maximal cliques.
• Each maximal clique of $G_T$ looks like this:
  - $\{v\} \cup N(v)$
• There are no other maximal cliques in $G_T$.
• If tree T is just a star (one vertex connected to all others), then $G_T$ is a simple clique.
• If $G_T$ is not a clique, then it has more than one maximal clique, and then T is not a star.
One more important property

• If \((x,y)\) is an edge of \(T\)
  - Then the vertices \(x\) and \(y\) appear together in exactly two maximal cliques, except if one of them is a leaf
• If one of them is a leaf, then they appear together in exactly one maximal clique
Properties of Cliques of $G_T$

- Vertex $v$ is present in $\leq \text{deg}(v)+1$ maximal cliques
  - $\text{Deg}(v)$ is degree of vertex $v$
- If $v$ has $k>0$ leaves as neighbors in $T$, then $v$ is present in exactly $\text{deg}(v) - k + 1$ maximal cliques
- If $v$ has $m$ non-leaves as neighbors in $T$, then $v$ is in
  - exactly $m + 1$ maximal cliques, if $v$ is not a leaf
- If $v$ has no leaves as neighbors in $T$, then $v$ is in
  - exactly $\text{deg}(v) + 1$ maximal cliques, if $v$ is not a leaf
- If $v$ is a leaf, it is in exactly 1 maximal clique
Algorithmic Ideas

1. Identify all maximal cliques of $G_T$
2. For each vertex $v$, compute
   - $C[v] = \#\text{ of maximal cliques of } G_T \text{ containing } v$
3. Identify leaves of $T$: all vertices with $C[v] = 1$
4. Figure out how many non-leaf neighbors each vertex has.
5. Figure out pairs of non-leaf vertices connected by an edge (present in exactly 2 max cliques)
More Properties of leaves of $T$

- If two leaves $x$ and $y$ are connected to the same non-leaf node, then they appear together in exactly one maximal clique and in no other clique.
- If two leaves $x$ and $y$ are not connected to the same non-leaf node, then they never appear together in a maximal clique.
Algorithmic Ideas

1. Figure out all leaves of T
2. Identify all edges of T connecting non-leaves (skeleton T’)
3. Figure out groups of leaves connected to same non-leaf
4. Figure out which leaf is connected to which non-leaf:
   a) Construct skeleton T’
   b) Construct maximal cliques of T’ corresponding to non-leaf
   c) Each maximal clique $A'$ of T’ corresponds to only one maximal clique $A$ of $G_T$ and to one non-leaf node $v$.
   d) Connect all leaf nodes in $A$ to non-leaf node $v$