Tree Augmentation

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Fall 2019

The Problem: CodeChef CHN15E

- Given tree T, the **augmented tree** G_T is defined as the graph obtained by joining every pair of vertices at distance 2 from each other.
- The problem is to construct T, given G_T.

Simple Properties

- Vertices of T and G_T are the same.
- Let neighbors of vertex v in T be the set N(v)
- The set {v} U N(v) forms a **clique** in G_T.
  - A subset of vertices in a graph forms a **clique** if all of them are connected by edges (i.e., no pair of vertices in this subset are missing an edge)
- A **maximal clique** is a set of vertices that forms a clique for which no superset is a clique.

More Properties

- For a tree T with n vertices, the **augmented tree** G_T has at most n maximal cliques
- Each maximal clique of G_T looks like this:
  - {v} U N(v)
- There are no other maximal cliques in G_T.
- If tree T is just a **star** (one vertex connected to all others), then G_T is a simple clique
- If G_T is not a clique, then it has more than one maximal clique, and then T is not a star.

One more important property

- If (x,y) is an edge of T
  - Then the vertices x and y appear together in exactly two maximal cliques, except if one of them is a leaf
- If one of them is a leaf, then they appear together in exactly one maximal clique

Properties of Cliques of G_T

- Vertex v is present in <= deg(v)+1 maximal cliques
  - Deg(v) is degree of vertex v
- If v has k>0 leaves as neighbors in T, then v is present in exactly deg(v) – k + 1 maximal cliques
- If v has m non-leaves as neighbors in T, then v is in
  - Exactly m + 1 maximal cliques, if v is not a leaf
- If v has no leaves as neighbors in T, then v is in
  - exactly deg(v) + 1 maximal cliques, if v is not a leaf
- If v is a leaf, it is in exactly 1 maximal clique

Algorithmic Ideas

1. Identify all maximal cliques of G_T
2. For each vertex v, compute
   - $C[v]$ = # of maximal cliques of G_T containing v
3. Identify leaves of T: all vertices with $C[v] = 1$
4. Figure out how many non-leaf neighbors each vertex has.
5. Figure out pairs of non-leaf vertices connected by an edge (present in exactly 2 max cliques)

More Properties of leaves of T

- If two leaves x and y are connected to the same non-leaf node, then they appear together in exactly one maximal clique and in no other clique
- If two leaves x and y are not connected to the same non-leaf node, then they never appear together in a maximal clique

Algorithmic Ideas

1. Figure out all leaves of T
2. Identify all edges of T connecting non-leaves (skeleton T')
3. Figure out groups of leaves connected to same non-leaf
4. Figure out which leaf is connected to which non-leaf:
   a) Construct skeleton T'
   b) Construct maximal cliques of T' corresponding to non-leaf
   c) Each maximal clique A' of T' corresponds to only one maximal clique A of G_T and to one non-leaf node v.
   d) Connect all leaf nodes in A to non-leaf node v