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Critical Sensor Density for Partial Connectivity in Large Area Wireless Sensor Networks

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***Abstract*—Assume sensor deployment follows the Poisson distribution. For a given partial connectivity requirement ρ, 0.5 < ρ < 1, we prove, for a hexagon model, that there exists a critical sensor density λ_0, around which the probability that at least $100\rho\%$ of sensors are connected in the network increases sharply from ε to $1-\varepsilon$ within a short interval of sensor density λ. The location of λ_0 is at the sensor density where the above probability is about 1/2. We also extend the results to the disk model. Simulations are conducted to confirm the theoretical results.**

I. Introduction

The problem of connectivity of sensors deployed randomly in a large area is a central issue in the studies of such networks. Each sensor connects only locally to the sensors within its communication range. The connections among the sensors across the field rely on intermediate relays. Extensive studies have been done on connectivity of sensor networks. Some early works focused on how many neighbors each node should connect to in order to maximize the network throughput, such as [1], [2], [3], [4]. More recent works studied how large the node transmission range is needed to ensure the full connectivity of the network (with high probability), or equivalently how many nodes are needed to ensure such connectivity given node transmission range is fixed [5], [6], [7], [8]. Some other works investigated the relationship between network connectivity and sensor coverage and studied the deployment of minimal number of sensors in a field such that the network is connected and the field is fully covered [9], [10], [11].

However, most of those studies focus only on the full connectivity (or k-connectivity) of the sensors in which every single sensor is connected to the entire network. This is a rather stringent requirement. To address the issue, we investigate the partial connectivity of the network in this paper. Our simulations show that the full coverage or connectivity would require much greater number of sensors in the field, just to avoid the exclusion of a small fraction of the isolated sensors. The savings from reducing the number of sensors will be significant if we are willing to give up a small portion of the sensors.

For a given percentage requirement ρ, we are interested in the events like "the percentage of the sensors that are connected is at least ρ". We want to find how, in general, the probability of this event is related to the sensor density. It is clear that a higher density will always increase the chance of the occurrence of such an event, and the question of our concern is how fast the chance will increase when the sensor density increases. Through our analysis, we find that there exists a critical value of the sensor density λ_0 such that the probabilities of the above events increase sharply from some small $\varepsilon > 0$ to $1-\varepsilon$ in a short interval centered near λ_0. Our analysis also reveals that, roughly speaking, the length of this interval is of the order $O(-\log\varepsilon/\log A)$, where A is the area of the field. We further find that λ_0 is such a value at which the probability for the event to occur is about 1/2 when A is large. More precise statements for these results are given in the next section for a hexagon model. We will also apply them to obtain some estimates for a disk model.

The concept of partial connectivity is not new [12]. Indeed, in the context of continuum percolation [13], the connectivity problem can be formulated using a percolation model in Poisson blobs. It was shown in [14], for example, that for a given sensor density above the percolation threshold, one can achieve a percentage of connectivity with a probability approaching to 1 exponentially fast, as the sensor field goes to the infinity. The percentage of the connectivity converges to the corresponding percolation probability, and the convergence rate depends implicitly on the given sensor density. This result is, however, different from ours. Here we focus on how fast the probability of a partial connectivity with a given connection percentage can get close to 1, from nearly 0, when the sensor area is large but fixed, and the sensor density is allow to change. This later question can be more important in an actual sensor deployment situation where a sensor area is pre-fixed and one needs to decide how many sensors to use. In this work we provide an answer to the later question by establishing a sharp-threshold property [15] around a "critical value" of the sensor density.

II. Properties of connectivity probabilities

This section contains all the main results. Due to the space limit, we omit the proofs of the theorems. In the next section we will use the simulations to illustrate these analytical results.

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Let λ be the sensor density, the number of sensors per unit area. Suppose N_A sensors are deployed randomly and independently in a square region of area A, where we assume N_A is a Poisson random variable $N_A \sim Poisson(\lambda A)$. Note that this is equivalent to assuming that the sensors are located according to a Poisson point process. Sensor networks are formed through communications between sensors. In a disk model, two sensor nodes can directly communicate with each other if they are in each other's communication range. We are interested in the probability that certain percentage of sensors are connected to each other in a network under a given density λ . A direct analysis of this probability for a given M is difficult. We are, however, able to study the problem in a slightly simplified model, i.e., the hexagon model. In what follows, we will first formulate the hexagon model and study the connectivity problem of the sensor network in this model. We will then use the results from the hexagon model to estimate the connectivity probability of the disk model.

A. The Hexagon Model

We partition the sensor field into hexagons using a regular honeycomb hexagon lattice. Hexagons are arranged in such a way that each hexagon has two of its six sides placed vertically. Hexagons with adjacent vertical sides line up to form a row. Suppose that in the lattice there are M rows and that in each row, there are M hexagons. Let $H_{i,j}$ denote the jth hexagon in the ith row, $i,\ j\ = 1, 2, \cdots, M$. Every off-boundary hexagon $H_{i,j}$ has six neighbors, including $H_{i,j-1}$ and $H_{i,j+1}$ on each side. Let $H = \{H_{i,j} :\ i,\ j\ = 1, 2, \cdots, M\}$. Note that for large M, $A = O(M^2)$.

Let A_H be the area of each hexagon. From the previous assumption, there is an equal probability for a sensor to fall into each $H_{i,j}$. The probability that a hexagon will contain at least one sensor depends on λ. We denote this probability as $p(\lambda)$. Then

$$p(\lambda) = 1 - e^{-\lambda A_H}. \tag{1}$$

We say a hexagon is occupied, if there is at least one sensor in it. Thus, $p(\lambda)$ is the probability that a hexagon is occupied when the sensor density is λ. We say two neighboring hexagons are directly-connected if they are both occupied. Two hexagons in the lattice are said to be connected, if there is a path of directly-connected hexagons that connects them. Hexagons that are connected with each other form a cluster (a maximal connected component) in the corresponding graph. In this model, sensors are divided into a set of disconnected clusters.

In the hexagon model, we assume that two sensors communicate with each other if and only if they are either in the same hexagon or in the neighboring hexagons. According to this definition of connectivity, two sensors are connected if and only if they are in the same cluster of hexagons. We are interested in the percentage of sensors which are connected within a given cluster. Let C be a cluster in the network, and N_C the total number of sensors in C. Let us consider the ratio

$$r_N(C) = \frac{N_C}{N_A}. \tag{2}$$

For a given ρ, $1/2 < \rho < 1$, we defined the event

$$B_\rho = \{\text{There is a cluster } C \text{ such that } r_N(C) \geq \rho\}. \tag{3}$$

B_ρ is the event that at least $100\rho\%$ of the sensors in the sensor field are connected. Note that since $\rho > 1/2$, there can be at most one such cluster. This is the largest cluster among all the clusters in the network. Let $P_\lambda(B_\rho)$ be the probability of event B_ρ under sensor density λ. We are interested in how $P_\lambda(B_\rho)$ changes as λ changes. To analyze this probability we first discuss some general properties of clusters of hexagons.

B. The largest cluster in the hexagon network

Here we first focus on a more coarse problem. For each hexagon, instead of counting how many sensors in it, we only ask if it is occupied. We then define an event similar to B_ρ as follows. Suppose C is a cluster of connected hexagons in the network. Let H_C be the total number of hexagons in cluster C. We define the relative size of C in terms of the relative area in the network:

$$r_H(C) = \frac{H_C}{M^2},$$

and define, for a given ρ, $1/2 < \rho < 1$, the event

$$D_\rho = \{\text{There is a cluster } C \text{ such that } r_H(C) \geq \rho\}. \tag{4}$$

Let $P_\lambda(D_\rho)$ be the probability of event D_ρ under sensor density λ. $P_\lambda(D_\rho)$ represents the probability that certain percentage of the whole sensor area is covered by the largest cluster of connected sensors.

To analyze this probability, we introduce a set of binary random variables $X = \{X_{i,j} :\ i,\ j = 1, \cdots, M\}$ such that $X_{i,j} = 1$, if $H_{i,j}$ is occupied and $X_{i,j} = 0$ otherwise. Then $X_{i,j},\ i,\ j = 1, \cdots, M$, are independent random variables with $P(X_{i,j} = 1) = 1 - P(X_{i,j} = 0) = p(\lambda)$. Let $\Omega = \{0, 1\}^{\{1,\cdots,M\}\times\{1,\cdots,M\}}$. A realization $x = \{x_{i,j} :\ i, j = 1, \cdots, M\} \in \Omega$ of X defines a network configuration. Ω is the configuration space of the network.The event D_ρ is completely determined by the realizations $x \in \Omega$ of X.

If we use the center of each hexagon to represent that hexagon, and declare that a center is "open" if the corresponding hexagon is occupied, and "close" if otherwise, then it is not difficult to see that our problem here is actually a site percolation problem in a triangular lattice. A well known result from the percolation theory asserts that there is a critical probability $p_0 = 1/2$, such

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that to have an infinite cluster in the lattice as $M \to \infty$, it is necessary and sufficient to have $p(\lambda) > p_0 = 1/2$. Since $p(\log 2/A_H) = 1/2$, an immediate consequence of this result is that if $\lambda < \log 2/A_H$ then $P_\lambda(D_\rho) \to 0$, as $M \to \infty$. It is also easy to see that for every $M < \infty$, $P_\lambda(D_\rho)$ is a differentiable and strictly increasing function of λ such that $P_0(D_\rho) = 0$ and $P_\infty(D_\rho) = 1$.

The following preliminary result is the key to the main result of this work.

Theorem 1. There is a constant $c > 0$, independent of M, and a $\lambda_0 = \lambda(A_H, M, \rho) > \log 2/A_H$ such that for all positive $\lambda \leq \lambda_0$,

$$P_\lambda(D_\rho) \leq \frac{1}{2} M^{-c[p(\lambda_0) - p(\lambda)]} \tag{5}$$

and, for all $\lambda \geq \lambda_0$, any small $\delta > 0$, and any small $\varepsilon_1 > 0$, there is an $M_0(\delta, \varepsilon_1)$ such that for all $M > M_0(\delta, \varepsilon)$,

$$P_\lambda(D_{\rho-\delta}) \geq 1 - \left(\frac{1}{2} + \varepsilon_1\right) \frac{1}{2} M^{-c[p(\lambda) - p(\lambda_0)]}. \tag{6}$$

This theorem is proved based on a general sharp-threshold inequality of increasing events in a lattice Bernoulli percolation model, which is a special case of the problem treated in [16]. We call λ_0 in Theorem 1 the critical density. Note that λ_0 depends on M, ρ and, in particularly, A_H.

C. Probability of connectivity

With a simple large deviation argument, Theorem 1 implies the following theorem for the connectivity probability $P_\lambda(B_\rho)$.

Theorem 2. Let λ_0 and constant c be as given in Theorem 1. For every fixed small δ and small ε_1, and for all sufficiently large M (depending on δ and ε_1),

$$P_\lambda(B_{\rho+\delta}) \leq \left(\frac{1}{2} + \varepsilon_1\right) M^{-c[p(\lambda_0) - p(\lambda)]}, \tag{7}$$

whenever $\lambda \leq \lambda_0$, and

$$P_\lambda(B_{\rho-\delta}) \geq 1 - \left(\frac{1}{2} + \varepsilon_1\right) M^{-c[p(\lambda) - p(\lambda_0)]}, \tag{8}$$

whenever $\lambda \geq \lambda_0$.

For small δ, if we agree that $P_\lambda(B_{\rho-\delta}) \approx P_\lambda(B_{\rho+\delta}) \approx P_\lambda(B_\rho)$, then Theorem 2 asserts basically the following. For sufficiently large M and some small $\varepsilon > 0$, if λ' and λ'' are such that $\lambda' < \lambda_0 < \lambda''$ and that $P_{\lambda'}(B_\rho) = \varepsilon$ and $P_{\lambda''}(B_\rho) = 1 - \varepsilon$, then distance between λ'' and λ' is

$$\lambda'' - \lambda' = O\left(-\frac{\log \varepsilon}{\log M}\right).$$

In other words, if M is sufficiently large, a relatively small increment in sensor density λ in a neighborhood of λ_0 can result a significant increase of the probability $P_\lambda(B_\rho)$. On the other hand, any change of values of λ outside this neighborhood will have much less significant influence. While the existence of λ_0 is established, its exact value is unknown. It should be determined experimentally.

It follows immediately from Theorem 2 that for any small $\delta > 0$,

$$\limsup_{M\to\infty} P_{\lambda_0}(B_{\rho+\delta}) \leq \frac{1}{2} \leq \liminf_{M\to\infty} P_{\lambda_0}(B_{\rho-\delta}), \tag{9}$$

which indicates, approximately, where the critical sensor density is located for large M.

D. Connectivity in the disk model

Now we discuss how to extend above results to the disk model. According to the definition of connectivity for a disk model, we can estimate the connectivity probability for the disk model with two hexagon models, H_1 and H_2 say, to obtain upper and lower bounds. This can be done as follows.

Let B_ρ^D be the event that at least $100\rho\%$ of the sensors are connected in a single cluster in the disk model. Suppose the transmission range for all sensors is R_c. In the first hexagon model H_1, we scale the size of the hexagons so that the farthest distance between points of two neighboring hexagons equals R_c. Then, a connection between any two hexagons in the lattice always implies a connection between any pair of sensors inside these two hexagons in the disk model. Let $B_\rho^{H_1}$ be the event in H_1 as defined in subsection A. Then, $B_\rho^{H_1} \subset B_\rho^D$ and $P(B_\rho^{H_1}) \leq P(B_\rho^D)$. Similarly, we can scale the size of the hexagons so that whenever the distance between two sensors is less or equal to R_c, these two sensors are either in the same hexagon or in two neighboring hexagons. We choose the smallest possible hexagon size for this to happen and define the corresponding lattice as H_2. Let $B_\rho^{H_2}$ be the corresponding connectivity event. Then $P(B_\rho^D) \leq P(B_\rho^{H_2})$.

Applying Theorem 2 to both $B_\rho^{H_1}$ and $B_\rho^{H_2}$, we conclude that there are $\lambda_1^{H_1}$ and $\lambda_2^{H_2}$ such that (7) holds for $B_\rho^{H_1}$ and $\lambda_1^{H_1}$, and (8) holds for $B_\rho^{H_2}$ and $\lambda_2^{H_2}$.

Theorem 3. There is a constant $c > 0$, independent of M, and two sensor densities λ_1^H and λ_2^H such that for every fixed small $\delta > 0$ and small ε_1, and for all sufficiently M,

$$P_\lambda(B_{\rho+\delta}) \leq \left(\frac{1}{2} + \varepsilon_1\right) M^{-c[p(\lambda_1^{H_1}) - p(\lambda)]}, \tag{10}$$

whenever $\lambda \leq \lambda_1^{H_1}$, and

$$P_\lambda(B_{\rho-\delta}) \geq 1 - \left(\frac{1}{2} + \varepsilon_1\right) M^{-c[p(\lambda) - p(\lambda_2^{H_2})]}, \tag{11}$$

whenever $\lambda \geq \lambda_2^{H_2}$.

III. SIMULATIONS

The purpose of the current simulations is twofold. First, we compare the connectivity probabilities of the hexagon model to the disk model. Second, we study how the connectivity probability, under various connectivity percentage ρ, is affected by the sensor density λ.

A. Methodology and Simulation Settings

The simulation code is written in $Matlab$. The sensor field is chosen to be a square region with an area of 50×50 units. We remark that this is actually an appropriate area, and we do not intend to use any larger sensor field in our simulations, because this study is not about a limiting property of sensor connectivity in which the area of the sensor field goes to infinity, but rather, it is about the “sharp-threshold" phenomenon. We are interested in sensor fields of any reasonable size (preferably, smaller sizes) at which the sharp-threshold phenomenon emerges. The number N_A of sensors follows a Poisson distribution with given density λ. Sensor nodes are randomly placed in the sensor field. All sensor nodes have a fixed transmission range R_c, which is set to 1. The size of the hexagons is scaled according to R_c to compare the connectivity properties between the hexagon model and the disk model.

The algorithm, as shown in Algorithm 1, takes the connectivity percentage ρ as input and outputs a sequence of pairs of λ and $P_\lambda(B_\rho)$ (the probability of meeting ρ at density λ). To compute $P_\lambda(B_\rho)$, we obtain a sufficiently large number n of network samples to get a stable estimate of $P_\lambda(B_\rho)$, which is the total number of network samples that meet percentage ρ divided by n. The plots in all simulation figures are the average of 100 runs.

Algorithm 1

Input: ρ.
Output: Sequence of pairs (λ, $P_\lambda(B_\rho)$).

```
λ = Min Value;
repeat
  cnt = 0;
  for i = 1 to n do
    Generate a Poisson number N_A with mean λ;
    Place N_A nodes randomly in the area;
    Obtain a network by connecting nodes using
    hexagon model (or disk model);
    if exist a cluster larger than 100ρ% then
      cnt++;
    end if
  end for
  P_λ(B_ρ)=cnt/n; output (λ, P_λ(B_ρ));
  λ+ = h; // h is a small increment of λ
until λ > Max Value
```

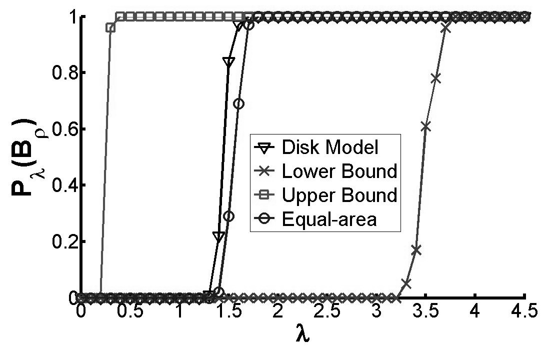

Figure 1. $P_\lambda(B_\rho)$ versus λ with different hexagon sizes

B. Hexagon Size in the Hexagon Model

In this set of simulations, we compare the connectivity probability of the disk model to that of the hexagon models. In the first hexagon model, we set the hexagons to be as large as possible, as long as any connected nodes in the hexagon model remain connected in the disk model. In the second hexagon model, we set the hexagons to be as small as possible size as long as the opposite connectivity property holds. We also study a third hexagon model, called an equal-area model, in which the size of the hexagon is such that the total area of any hexagon along with its 6 neighbors equals the area covered by a disk with radius R_c. Figure 1 shows the comparisons, where the first two hexagon models provide the bounds for the connectivity probability for the disk model and the third hexagon model shows a closer match to the disk model.

We observe the following. 1) The hexagon model with various hexagon sizes exhibits the same pattern as the disk model. There is always a sharp increase of $P_\lambda(B_\rho)$ from 0 to 1 near some critical density λ_0. 2) The values of λ_0 for different size of hexagons have a large variance. As the hexagon size increases, λ_0 decreases from 3.5 (for lower bound of hexagon size) to 0.2 (for upper bound of hexagon size). This is because for a larger hexagon size, there is a higher chance for nodes to fall into the same hexagon or neighboring hexagons resulting a higher probability of network connection. That is, to reach the same $P_\lambda(B_\rho)$, the hexagon model with a larger hexagon size requires less node density, leading to a smaller λ_0. 3) The critical density value of the equal-area hexagon model is very close to the disk model. For the rest of simulation, we use the hexagon size of equal-area for the hexagon model.

C. Sharp-threshold Property near Critical Density

In the simulations we record the relations between the connectivity probability and the number of sensors deployed in the network. For $\rho = .95$, we observe for the disk model that, when the number of sensors is around 4127 ($\lambda = 1.61$), the connectivity probability $P_\lambda(B_\rho)$ is

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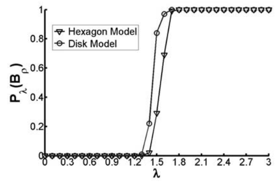
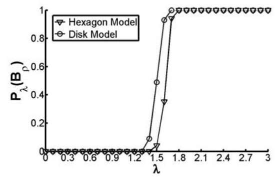
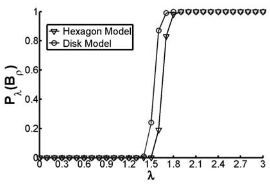


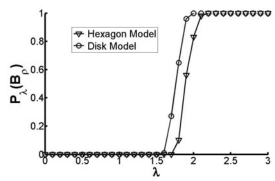
(a) ρ = 0.5



(b) ρ = 0.65



(c) ρ = 0.75



(d) ρ = 0.95

Figure 2. $P_\lambda(B_\rho)$ versus λ with different ρ.

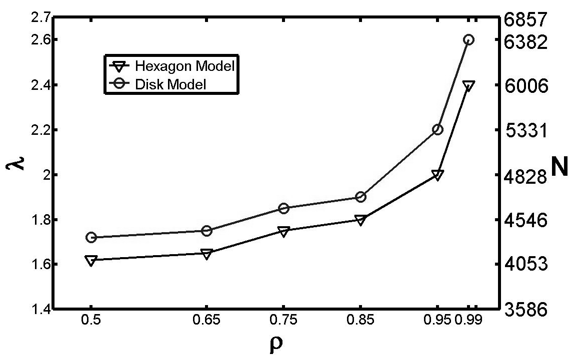


Figure 3. ρ versus λ when $P_\lambda(B_\rho) \geq 0.99$

almost 0; but when the number of sensors changes to 4914 ($\lambda = 1.95$), a less than 20% increase in the number of sensors, the probability $P_\lambda(B_\rho)$ increases to almost 1. Similar sharp increase in connectivity probability is also seen for the hexagon model. Simulations show that in the hexagon model when the number of sensors goes from 4409 ($\lambda = 1.75$) to 5408 ($\lambda = 2.14$), a 22% increase in the number of sensors, the probability $P_\lambda(B_\rho)$ goes up from nearly 0 to almost 1. These results suggest that it is important to find the critical density for large scale deployment of sensors. It helps us determine the right number of sensors to deploy. With a slight increase in node density over this critical point, the connectivity probability can be increased significantly.

D. Impact of Parameter ρ

In this set of simulations, we investigate the impact of the parameter ρ. Figure 2 shows a group of charts of $P_\lambda(B_\rho)$ versus λ where ρ varies from 0.5 to 0.95. All charts for different ρ exhibit the same pattern of the increase of $P_\lambda(B_\rho)$ as λ increases. It suggests that the increasing rate of $P_\lambda(B_\rho)$ is independent of ρ, at least for ρ in a range bounded away from 1.

Figure 3 shows the correlation between ρ and λ when $P_\lambda(B_\rho) \geq 0.99$ (i.e., highly sure that $100\rho\%$ of nodes are connected). We see that in general the increase of λ with ρ is slow for small ρ. But, the increase of λ becomes sharper for large ρ. This is particularly true when ρ reaches 95% or beyond.

Figure 3 also translates the density λ into the number of sensor nodes (the right hand side vertical bar). We see that when ρ is increased from 0.95 to 0.99, the number of sensors required to meet the connectivity percentages jumps significantly from 4828 to 6006, which will be much higher if ρ is further increased. To attain the full connectivity, the number of sensors will be prohibitly large. This result justifies the need for the study of partial connectivity. It shows that it is much more economic to sacrifice a small percentage of node connection and satisfy with a partial (but with high percentage) connectivity. This way we can save a large number of sensor nodes.

IV. Acknowledgment

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