Introduction to Quantum Computing

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Outline

Quantum Mechanics

What is quantum mechanics?

Quantum mechanics is a mathematical framework for developing physical theories.

How does the mathematical framework connect to the physical world?

The postulates of quantum mechanics lead to mathematical models of the physical world. Experimental results verifying the predictions of the models have so far held up the validity of the postulates.

What are the postulates of quantum mechanics?

State Space Postulate of Quantum Mechanics

State Space Postulate: Every closed (isolated) physical system is associated with a Hilbert space $\mathcal{H}$ (complex vector space with inner product) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the state space.

- $\mathcal{H}$ can be finite or infinite dimensional.
  - To describe the state of a particle that is free to occupy any point in some region of space requires a continuous Hilbert space.
  - To describe realistic quantum computing models, finite-dimensional Hilbert space would suffice.
  - We would only focus on systems that are composed of two-level systems.

Qubits

- Qubits are encoded using the states in a two-level system, which are described by vectors in a 2-dimensional Hilbert space.
- Computational basis: We choose two orthonormal (orthogonal to each other and with unit 2-norm) basis vectors and call them $|0\rangle$ and $|1\rangle$.
- Hadamard basis: Another useful basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Qubits

- Examples of two-level systems
    - A single photon that can be found in one of two paths; A single photon that is either present or absent at a particular location
    - Spin state of particles
    - Energy level of an electron orbiting a nucleus.
- Analogous to the classical bit.

Qubits

- What is different about qubits?
- Qubit can assume any superposition (linear combination) of its basis states.
- General state of a two-level system is described by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for complex α and β.
- We choose the following coordinate system for convenience.

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- α and β are the probability amplitudes and $|\alpha|^2 + |\beta|^2 = 1$.
- $|\alpha| = +\sqrt{\alpha\alpha^*}$ is the magnitude (also, absolute value) of the complex number α.
- Note that $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ are equivalent. θ is called the phase and $e^{i\theta}$ is called the phase factor.

Qubits

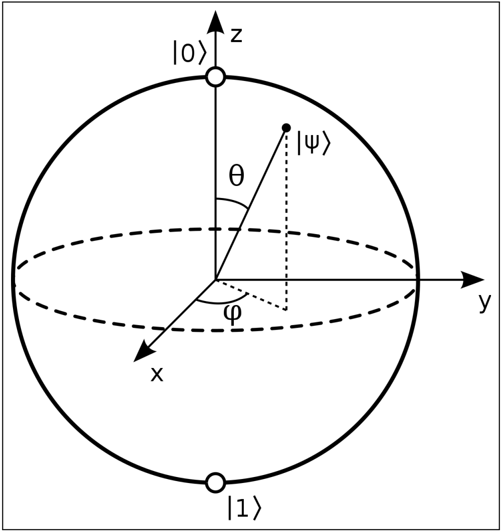
- $\alpha |0\rangle + \beta |1\rangle$ is distinct from $\alpha |0\rangle + e^{i\theta}\beta |1\rangle$.
- $e^{i\theta}$ is called the relative phase factor.
- $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ can also be expressed as

$$|\psi\rangle = \cos(\frac{\theta}{2}) |0\rangle + e^{i\phi} \sin(\frac{\theta}{2}) |1\rangle$$

where

- $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ is a point on the 3-dimensional unit sphere (Bloch sphere).

Bloch Sphere



Time-Evolution of States Postulate

The time-evolution of an isolated quantum system is described by a unitary operator.

- For any evolution of the system starting in state $|\psi_1\rangle$ and in state $|\psi_2\rangle$ after the evolution, there exists a unitary operator U such that

$$|\psi_2\rangle = U\,|\psi_1\rangle$$

- U acting on a single qubit is called 1-qubit (unitary) gate.
- A unitary operator U is a linear operator such that $U^\dagger U = UU^\dagger = I$.
- Rows (columns) have unit norm and are orthogonal to each other.
- The evolution is reversible since $U^{-1} = U^\dagger$ is unitary.

1-qubit Gates

- 1-qubit gates can be represented as 2×2 matrices.
- In the computational basis, 1-qubit operator U can be represented as

$$\begin{bmatrix} U\,|0\rangle & U\,|1\rangle \end{bmatrix}$$

- For example, NOT gate is represented as

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as it sends $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

1-qubit Gates

- $$\mathrm{NOT}\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \text{ and}$$
  $$\mathrm{NOT}\,|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$
- NOT gate's action on every other state is determined by linearity. For example,
  $$\mathrm{NOT}(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\mathrm{NOT}\,|0\rangle + \beta\mathrm{NOT}\,|1\rangle$$

Pauli Gates

- The NOT gate is also called an X gate and is one of the four Pauli matrices.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Every 1-qubit gate can be expressed as a linear combination of the Pauli gates.

Other Important 1-qubit Gates

- Hadamard, phase, $\pi/8$ gates:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \qquad = e^{i\pi/8}\begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$$

Rotation Operators

$$
\begin{aligned}
R_x(\theta) = & e^{-i\theta X/2} = \quad \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\
R_y(\theta) = & e^{-i\theta Y/2} = \quad \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\
R_z(\theta) = & e^{-i\theta Z/2} = \quad \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}
\end{aligned}
$$

Canonical Representation of 1-qubit Gates

Theorem

For every unitary 1-qubit operator U, there exists $\alpha, \beta, \gamma,$ and δ such that

$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_z(\delta)$$

Composition of Systems Postulate

When two physical systems described by $\mathcal{H}_1$ and $\mathcal{H}_2$ are treated as one combined system, the state space of the combined system is the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$.

- The state space of a system composed of n distinct subsystems is the tensor product of state spaces of the n subsystems.
- If each subsystem is of dimension 2, the n-fold tensor product space is of dimension 2^n.

Entanglement

- In a system composed of two subsystems, if the first system is in state $|\psi_1\rangle$ and the second system in state $|\psi_2\rangle$, then the combined system is in state

$$|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle \, |\psi_2\rangle = |\psi_1 \psi_2\rangle$$

- The systems in this case are isolated from each other and do not interact with each other.
- However, the state of two interacting subsystems cannot always be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. In this case, we say that the two systems are entangled.
- The state of the composite system can in general be a linear combination of the vectors of the type $|\psi_1\rangle \otimes |\psi_2\rangle$.
- Example of a 2-qubit entangled state: $|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$.

Unitary Operators for Composite Systems

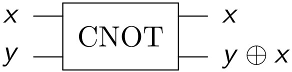
- In a 2-qubit system, let $|\psi_1\rangle$ be the state of the first qubit and the $|\psi_2\rangle$ that of the second qubit.
- If we apply the NOT gate X to the first bit and implicitly the gate I to the second qubit, we will end up in a state $X|\psi_1\rangle \otimes I|\psi_2\rangle = X \otimes I(|\psi_1\psi_2\rangle)$
- The unitary operator for the composite system

$$X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I = \begin{bmatrix} 0.I & 1.I \\ 1.I & 0.I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Although $X \otimes I$ is a 2-qubit operator, it is essentially a 1-qubit operator.
- There are 2-qubit gates which are not the tensor product of two 1-qubit gates.

CNOT Gate

- The classical definition of CNOT gate with inputs x and y is



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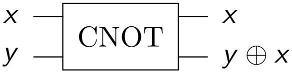
$$\begin{aligned}
\mathrm{CNOT}\,|00\rangle &\to |00\rangle \\
\mathrm{CNOT}\,|01\rangle &\to |01\rangle \\
\mathrm{CNOT}\,|10\rangle &\to |11\rangle \\
\mathrm{CNOT}\,|11\rangle &\to |10\rangle
\end{aligned}$$

CNOT Gate

- In matrix form,

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



- CNOT operator cannot be expressed as the tensor product of lower dimensional unitary operators.

Bell Basis

- The set of Bell states is an orthonormal basis for a 2-qubit system.

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\beta_{01}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\beta_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Measurement Postulate

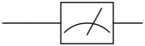
For a given orthonormal basis $B = \{|\phi_i\rangle\}$ of a state space $\mathcal{H}_A$ for a system A, it is possible to perform Von-Neumann measurement on A with respect to the basis B that, given a state

$$|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle$$

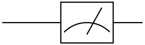
outputs a label i with probability $|\alpha_i|^2$ and leaves the system in the state $|\phi_i\rangle$.

- Evolution of the state of a system during a measurement is not unitary.
- States $|\psi\rangle$ and $e^{i\theta} |\psi\rangle$ are equivalent.
- We can implement Von-Neumann measurement in an arbitrary basis.

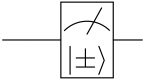
Measurement Postulate



$\frac{1}{2}(|0\rangle + |1\rangle)$



$|1\rangle$



Measuring a Subsystem

- Let $|\phi\rangle = \sqrt{\frac{1}{11}}\,|00\rangle + \sqrt{\frac{5}{11}}\,|01\rangle + \sqrt{\frac{2}{11}}\,|10\rangle + \sqrt{\frac{3}{11}}\,|11\rangle$
- If we measure the 2-qubit system with respect to the computational basis, the first state gets the label 0 with probability $\frac{6}{11}$.
- If we measure only the first qubit, we get the label 0 with probability $\frac{6}{11}$ and the system will be in the state

$$|0\rangle\,(\sqrt{\frac{1}{6}}\,|0\rangle + \sqrt{\frac{5}{6}}\,|1\rangle)$$

- and get the label 1 with probability $\frac{5}{11}$ and the system will be in the state

$$|1\rangle\,(\sqrt{\frac{2}{5}}\,|0\rangle + \sqrt{\frac{3}{5}}\,|1\rangle)$$

Linear Algebra

- Hilbert spaces
- Dual vectors
- Unitary operators
- Tensor products
- Spectral theorem

Hilbert Space

- A Hilbert space $\mathcal{H}$ is a vector space with an inner product.
- Denote the inner product of two vectors $\mathbf{v}$ and $\mathbf{w}$ from $\mathcal{H}$ by $\langle \mathbf{v}, \mathbf{w} \rangle$ which satisfies the following properties.
- Linearity in the second argument

$$\langle \mathbf{v}, \sum_i \lambda_i \mathbf{w}_i \rangle = \sum_i \lambda_i \langle \mathbf{v}, \mathbf{w}_i \rangle$$

- Conjugate-commutativity

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle^*$$

- Non-negativity

$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$$

with equality if and only if $\mathbf{v} = \mathbf{0}$.

Inner Product

- Example: The dot product of two vectors **v** and **w** from $\mathcal{H}$ is defined as

$$\mathbf{v}.\mathbf{w} = \mathbf{v}^{\dagger}\mathbf{w} = \sum_{i} {v_i}^{*} w_i$$

- $\langle\psi, \phi\rangle$ is written as $\langle\psi||\phi\rangle = \langle\psi|\phi\rangle$

Orthogonality and Norm

- Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if their inner product $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.
- The norm $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
- A vector is a unit vector if its norm is 1.
- A set of unit vectors which are mutually orthogonal is called an orthonormal set.
- If $\{|\psi_i\rangle\}$ is an orthonormal basis for $\mathcal{H}$, then every $|\phi\rangle$ can be written as

$$|\phi\rangle = \sum_i \langle \psi_i | \phi \rangle \, |\psi_i\rangle$$

Dual Space

The space $\mathcal{H}^*$ of linear functions that map $\mathcal{H}$ to $\mathbb{C}$ is called the dual space of $\mathcal{H}$.

- $\mathcal{H}^*$ is indeed a vector space.
- Quantities versus prices; position vectors versus force
- If $\{\mathbf{v}_i\}$ is a basis for $\mathcal{H}$, we define the dual basis $\{\mathbf{v}_i\}$ as follows. Set

  $$\mathbf{f}_i(\mathbf{v}_j) = \delta_{ij}$$

  and then extend them linearly.
- Dual basis as defined is indeed a basis for $\mathcal{H}^*$.
- The dimensions of $\mathcal{H}$ and $\mathcal{H}^*$ are the same if $\mathcal{H}$ is finite dimensional. Furthermore, $\mathcal{H}$ and $\mathcal{H}^*$ are isomorphic.
- For $\mathbf{v} \in \mathcal{H}$, if $\mathbf{f}(\mathbf{v}) = 0$ for all $\mathbf{f} \in \mathcal{H}^*$, then $\mathbf{v} = \mathbf{0}$.

Dual of the Dual Space

$\mathcal{H}^{**}$ is called the dual of the dual space $\mathcal{H}^{*}$.

- While $\mathcal{H}$ and $\mathcal{H}^{*}$ are isomorphic, there is no natural isomorphism between them for general vector spaces.
- $\mathcal{H}$ and $\mathcal{H}^{**}$ are naturally isomorphic. Let

$$\mathrm{eval}(\mathbf{v})(\mathbf{f}) = \mathbf{f}(\mathbf{v})$$

- eval establishes an isomorphism between $\mathcal{H}$ and $\mathcal{H}^{**}$.
- $\mathcal{H}$ and $\mathcal{H}^{**}$ are not necessarily isomorphic for infinite dimensional spaces.

Isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$

- There is no natural isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$ for general vector spaces.
- However, for inner product spaces, we have the isomorphism $S : \mathcal{H} \to \mathcal{H}^*$ between $\mathcal{H}$ and $\mathcal{H}^*$.

$$S(\mathbf{v}) = \langle \mathbf{v}, . \rangle$$

- $S(\mathbf{v})$ is the dual vector of $\mathbf{v}$
- $S(\mathbf{v})$ is the linear function that maps the vector $\mathbf{w} \in \mathcal{H}$ to $\langle \mathbf{v}, \mathbf{w} \rangle$.
- One of the main conceptual uses of inner product.
- Natural means without a choice of a basis.

Dual Vector Notation in Quantum Mechanics

- The dual vector of $|\phi\rangle$ is written as $\langle\phi|$ in quantum mechanics.
- $\langle\phi|$ maps $|\psi\rangle$ to $\langle\phi|\psi\rangle$.
- In matrix representation, $\langle\psi|$ is the conjugate transpose of the matrix (column vector) representing $|\psi\rangle$. For this purpose, we rely on the computational basis and the corresponding dual basis.

Example

- Let $|\psi\rangle = \sqrt{\frac{2}{3}}\,|01\rangle + \frac{i}{\sqrt{3}}\,|11\rangle$ and $|\phi\rangle = \sqrt{\frac{1}{2}}\,|10\rangle + \sqrt{\frac{1}{2}}\,|11\rangle$.
- As column vectors,

$$|\psi\rangle = \begin{bmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ \frac{i}{\sqrt{3}} \end{bmatrix} \text{ and } |\phi\rangle = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix}$$

- $$\langle\psi|\phi\rangle = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}} & 0 & -\frac{i}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix} = \frac{-i}{\sqrt{6}}$$

Linear Operators

- A (linear) operator T on a vector space $\mathcal{H}$ is a linear transformation $T : \mathcal{H} \to \mathcal{H}$.
- The outer product $|\psi\rangle\langle\phi|$ of a vector $|\psi\rangle$ with a vector $|\phi\rangle$ is an operator with the action

$$|\psi\rangle\langle\phi|\,|\gamma\rangle = |\psi\rangle\langle\phi|\gamma\rangle\,\langle\phi|\gamma\rangle\,|\psi\rangle$$

- If $\{|b_i\rangle\}$ is an orthonormal basis for a vector space $\mathcal{H}$, then every operator on $\mathcal{H}$ can be written as

$$T = \sum_i \sum_j T_{i,j}\,|b_i\rangle\langle b_j|$$

  where $T_{i,j} = \langle b_i|T|b_j\rangle$.
- $\{|b_i\rangle\langle b_j|\}$ is a basis for the space of operators on $\mathcal{H}$.

$$T\,|\psi\rangle = \sum_i \sum_j T_{i,j}\,|b_i\rangle\langle b_j|\,|\psi\rangle \sum_i \sum_j T_{i,j}\,\langle b_j|\psi\rangle\,|b_i\rangle$$

Special Operators

- An operator U is unitary if $U^* = U^{-1}$, that is, $U^*U = UU^* = I$.
- An operator T is alertHermitian (or self-adjoint) if $T^* = T$.
- An operator P is a projector if $P^2 = P$.
- A projector P is an orthogonal projector if $P^* = P$.
- $|\psi\rangle \langle\psi|$ is an orthogonal projector for unit vectors $|\psi\rangle$.
- Resolution of identity in basis $\{|b_i\rangle\}$:

$$I = \sum_i |b_i\rangle \langle b_i|$$

Dual Operators

- Let $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear map between two Hilbert spaces. The dual S^* of S is defined as

$$S^*(\mathbf{g}) = \mathbf{g}S$$

- S^* sends a linear functional $\mathbf{g}$ on W to the operator $\mathbf{g}S$ which is a linear functional on V.
- Let M_S be the matrix representation of S in the orthonormal basis $\{\mathbf{v}_i\}$. Then $M_{S^*} = M_S^\dagger$ in the dual basis $\{\mathbf{f}_i\}$.

Tensor Products

The tensor product of two vector spaces $\mathcal{H}_m$ (with $\{|b_i\rangle\}$ as an orthonormal basis) and $\mathcal{H}_n$ (with $\{|c_j\rangle\}$ as an orthonormal basis) of dimensions *m* and *n* respectively is the *mn*-dimensional space $\mathcal{H}_1 \otimes \mathcal{H}_2$ spanned by the orthonormal basis $|b_i\rangle \otimes |c_j\rangle = |b_i\rangle |c_j\rangle$ subject to the conditions

- For any $c \in \mathbb{C}$, $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$

$$c(|\psi_1\rangle \otimes |\psi_2\rangle) = (c\,|\psi_1\rangle) \otimes |\psi_2\rangle = |\psi_1\rangle \otimes (c\,|\psi_2\rangle)$$

- For any $|\psi_1\rangle, |\phi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$

$$(|\psi_1\rangle + |\phi_1\rangle) \otimes |\psi_2\rangle = (|\psi_1\rangle \otimes |\psi_2\rangle) + (|\phi_1\rangle \otimes |\psi_2\rangle)$$

- For any $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle, |\phi_2\rangle \in \mathcal{H}_2$

$$|\psi_1\rangle \otimes (|\psi_2\rangle + |\phi_2\rangle) = (|\psi_1\rangle \otimes |\psi_2\rangle) + (|\psi_1\rangle \otimes |\phi_2\rangle)$$

Tensor Products

- Let A and B be operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $A \otimes B$ is an operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined by

$$(A \otimes B)(|\psi_1\rangle \otimes |\psi_2\rangle) = A\,|\psi_1\rangle \otimes B\,|\psi_2\rangle$$

  and extended linearly.

- $$(A \otimes B)(\sum_{i,j} \lambda_{i,j}\,|\psi_i\rangle \otimes |\phi_j\rangle) = \sum_{i,j} \lambda_{i,j} A\,|\psi_i\rangle \otimes B\,|\phi_j\rangle$$

Tensor Products in Matrix Notation

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix}$

then

$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & \cdots & a_{1n}b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & \cdots & a_{1n}b_{pq} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & \cdots & a_{mn}b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{b1} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & \cdots & a_{mn}b_{pq} \end{bmatrix}$$

Tensor Product in Matrix Notation

- Also

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

- A special case

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$

Conclusion

- Thank you